Exact soliton solutions of coupled nonlinear Schrödinger equations: Shape-changing collisions, logic gates, and partially coherent solitons

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(Received 16 May 2002; revised manuscript received 31 October 2002; published 25 April 2003)

The different dynamical features underlying soliton interactions in coupled nonlinear Schrödinger equations, which model multimode wave propagation under varied physical situations in nonlinear optics, are studied. In this paper, by explicitly constructing multisoliton solutions (up to four-soliton solutions) for two-coupled and arbitrary *N*-coupled nonlinear Schrödinger equations using the Hirota bilinearization method, we bring out clearly the various features underlying the fascinating shape changing (intensity redistribution) collisions of solitons, including changes in amplitudes, phases and relative separation distances, and the very many possibilities of energy redistributions among the modes of solitons. However, in this multisoliton collision process the pairwise collision nature is shown to be preserved in spite of the changes in the amplitudes and phases of the solitons. Detailed asymptotic analysis also shows that when solitons undergo multiple collisions, there exists the exciting possibility of shape restoration of at least one soliton during interactions of more than two solitons represented by three- and higher-order soliton solutions. From an application point of view, we have shown from the asymptotic expressions how the amplitude (intensity) redistribution can be written as a generalized linear fractional transformation for the *N*-component case. Also we indicate how the multisolitons can be reinterpreted as various logic gates for suitable choices of the soliton parameters, leading to possible multistate logic. In addition, we point out that the various recently studied partially coherent solitons are just special cases of the bright soliton solutions exhibiting shape-changing collisions, thereby explaining their variable profile and shape variation in collision process.

DOI: 10.1103/PhysRevE.67.046617 PACS number(s): 42.81.Dp, 42.65.Tg, 05.45.Yv

I. INTRODUCTION

The study of coupled nonlinear Schrödinger (CNLS) equations is receiving a great deal of attention in recent years due to their appearance as modeling equations in diverse areas of physics such as nonlinear optics [1], including optical communications [2], biophysics [3], multicomponent Bose-Einstein condensates at zero temperature [4], etc. To be specific, soliton type pulse propagation in multimode fibers [1] and in fiber arrays [5] is governed by a set of *N*-CNLS equations which is not integrable in general. However, it becomes integrable for a specific choice of parameters [6,7]. On the other hand, the recent studies on the coherent [8] and incoherent [9] beam propagation in photorefractive media, which can exhibit high nonlinearity with extremely low optical power, necessitate intense study of CNLS equations both integrable and nonintegrable. The first experimental observation of the so-called partially incoherent solitons with the excitation of a light bulb in a photorefractive medium [10] has made this study even more interesting. In this context of beam propagation in a Kerr-like photorefractive medium, the governing equations are a set of *N*-CNLS equations [11,12].

We consider the following *N*-CNLS equations of the Manakov type [13] for our study:

$$iq_{jz}+q_{jtt}+2\mu\sum_{p=1}^{N}|q_p|^2q_j=0, \quad j=1,2,\ldots,N, \tag{1}$$

where q_j is the envelope in the jth mode, z and t represent the normalized distance along the fiber and the retarded time, respectively, in the context of soliton propagation in multimode fibers. In the case of fiber arrays q_j corresponds to the jth core. Here 2μ gives the strength of the nonlinearity. In the framework of N self-trapped mutually incoherent wave packets propagation in Kerr-like photorefractive media [11,12], q_j is the jth component of the beam, z and t represent the coordinates along the direction of propagation and the transverse coordinate, respectively. The interesting property of the *N*-CNLS equations of form (1) is that they are integrable equations and possess soliton solutions.

It is obvious from Eq. (1) that for $N=1$ it corresponds to the standard envelope soliton possessing the integrable nonlinear Schrödinger equation, governing intense optical pulse propagation through a single mode optical fiber [1,14]. For the $N=2$ case, it reduces to the celebrated Manakov model [13] describing intense electromagnetic pulse propagation in birefringent fiber. Manakov himself has made a detailed asymptotic analysis of the inverse scattering problem associated with system (1) for $N=2$ and identified changes in the polarization vector [13]. However, no explicit two-soliton expression was given there. Very recently, Radhakrishnan, Lakshmanan, and Hietarinta have obtained the bright one- and two-soliton solutions for this case [15], and have revealed certain different shape changing (intensity redistribution) collision properties. These Manakov solitons have been observed recently in $Al_xGa_{1-x}As$ planar waveguides [16] and precisely this kind of energy exchanging (shape changing) collisions has been experimentally demonstrated in Ref. [17]. The results of Ref. [15] have led Jakubowski, Steiglitz,

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and Squier [18] to express the energy redistributions as linear fractional transformations so as to construct logic gates. Later, Steiglitz [19] explicitly constructed such logic gates including the universal NAND gate, based on the shape-changing collision property, and hence pointed out the possibility of designing an all optical computer equivalent to a Turing machine, at least in a mathematical sense. However, results are scarce for $N \geqslant 2$ case of Eq. (1) though they are of considerable physical importance as mentioned earlier.

The shape-changing collision property exhibited by the 2-CNLS equations, which has not been observed, in general, in any other simpler $(1+1)$-dimensional integrable system, requires a detailed analysis to identify the various possibilities and the underlying potential technological applications. In a very recent letter [20], the present authors have studied the multicomponent N-CNLS equations and shown that shape-changing collisions occur here also with more possibilities of energy redistribution. It has also been briefly pointed out that the much discussed partially coherent solitons (PCSs) [11,12], which are of variable shape, namely, 2-PCS, 3-PCS, . . . , N-PCS, are special cases of the two-soliton, three-soliton, . . . , N-soliton solutions of the 2-CNLS, 3-CNLS, . . . , N-CNLS equations, respectively. The understanding of variable shapes [11,12] of these recently experimentally observed partially coherent solitons [21] in photorefractive medium and their interesting collision behavior will be facilitated by obtaining the higher-order soliton solutions of the 2-CNLS and the N-CNLS ($N \geqslant 2$) equations.

In this paper, we wish to undertake a detailed analysis of the dynamical features associated with soliton interactions in multicomponent N-CNLS equations. There exist numerous interesting phenomena which one has to pay attention to in order to realize the full potentialities of these equations and the underlying different soliton dynamics. Some of the important aspects include the following among others.

(1) Explicit expressions for multisoliton solutions in multicomponent CNLS equations useful for analysis of interactions (as against formal expressions).

(2) Different soliton interactions involving shape-changing collisions.

(3) Dependence of shape changes and relative separation distances on amplitudes of the colliding solitons.

(4) Identification of different possibilities of energy redistributions among the different modes of the soliton during collision and obtaining generalized linear fractional transformations.

(5) State restoring properties in multisoliton solutions.

(6) Existence of partially coherent solitons, stationary and moving, as special cases of the above multisoliton solutions.

(7) Identification of multisoliton solutions as logic gates in multicomponent CNLS equations.

The present paper will be essentially devoted to the understanding of multisoliton interactions in N-CNLS equations, and its application in constructing logic gates and in identifying partially coherent solitons as special cases of multisoliton solutions. In particular, in the present paper, we will deduce explicit expressions for multisoliton solutions (up to four-soliton solutions), which can be easily generalized to the arbitrary soliton case, 2-CNLS and then for arbitrary N-CNLS equations. To start with, we will briefly consider the two-soliton solution to bring out the shape-changing nature of soliton collisions, which can be quantified in terms of generalized linear fractional transformations (LFTs), and identify the changes in amplitudes, phases, and relative separation distances among the solitons by carrying out appropriate asymptotic analysis. However, the standard (shape preserving) elastic collisions can occur for specific choice of soliton parameters (initial conditions). More interestingly, we also point out that when more than two solitons collide successively, say three solitons, there exists the exciting possibility of restoration of the shape of one of the three solitons leaving the other two undergoing shape changes and we prove that the underlying soliton interaction is pairwise. We give explicit conditions for the shape restoring property. Extending this analysis, one can easily check that in an M-soliton collision, it is possible to restore the states of $(M-2)$ solitons after collisions. Such possibilities lead to the construction of optical logic gates of different types and generalized linear fractional transformations, as we will show in this paper.

This paper is organized as follows. In Sec. II we briefly present the bilinearization procedure for the N-CNLS equations. Explicit multisoliton solutions (up to four) of the 2-CNLS equations are obtained in Sec. III. Then the generalization of these multisoliton solutions to N-CNLS equations is given in Sec. IV. The two-soliton collision properties of 2-CNLS and their generalization to N-CNLS equations are studied in Sec. V. In Sec. VI, we present a systematic procedure to identify the intensity redistribution among N modes in terms of a generalized linear fractional transformation which is the precursor to the development of logic gates without interconnecting discrete components [18]. The interesting features of the higher-order soliton solutions, namely, the pairwise nature of collision of solitons, and the shape restoration property of the state of one soliton only in a three-soliton collision process are presented in Sec. VII. In Sec. VIII we introduce the possibility of looking at the bright soliton solutions as logic gates, as an alternate point of view. Then in Sec. IX we demonstrate explicitly that for specific choices of the parameters of the bright soliton solutions, various PCSs reported in the literature result. The collision properties of PCSs and the salient features of multisoliton complexes are also discussed. Section X is allotted for a conclusion. Also in the Appendix we present the explicit form of the four-soliton solution.

II. BILINEARIZATION

The set of N-CNLS equations (1) has been found to be completely integrable [6,7] and admits exact bright soliton solutions. Their explicit forms can be obtained by using Hirota's bilinearization method [22], which is straightforward. Any of the other soliton producing methodologies in principle is equally applicable; however, this paper is not concerned with the relative merits of the various methodologies.

To start with, we make the bilinearizing transformation (which can be identified systematically from the Laurent expansion [6])

$$q_j = \frac{g^{(j)}}{f}, \quad j = 1,2, \ldots, N \tag{2}$$

to Eq. (1). This results in the following set of bilinear equations:

$$(iD_z + D_t^2) g^{(j)} f = 0, \quad j = 1,2, \ldots, N, \tag{3a}$$

$$D_t^2 ff = 2\mu \sum_{n=1}^{N} g^{(n)} g^{(n)*}, \tag{3b}$$

where * denotes the complex conjugate, $g^{(j)}$'s are complex functions, while $f(z,t)$ is a real function and Hirota's bilinear operators D_z and D_t are defined by

$$D_z^n D_t^m (ab) = \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m a(z,t) b(z',t') \Big|_{(z=z',t=t')} . \tag{3c}$$

The above set of equations can be solved by introducing the following power series expansions for $g^{(j)}$'s and f:

$$g^{(j)} = \chi g_1^{(j)} + \chi^3 g_3^{(j)} + \cdots, \quad j = 1,2, \ldots, N, \tag{4a}$$

$$f = 1 + \chi^2 f_2 + \chi^4 f_4 + \cdots, \tag{4b}$$

where χ is the formal expansion parameter. The resulting set of equations, after collecting the terms with the same power in χ, can be solved recursively to obtain the forms of $g^{(j)}$'s and f. Though a formal closed form solution of the N-soliton expression of Eq. (1) as a ratio of two $(N\times N)$ determinants can be given [23], it becomes necessary to deduce the explicit expressions (which is nontrivial) in order to understand the interaction properties at least for the lower-order solitons. In the following section we will only present the minimum details.

III. MULTISOLITON SOLUTIONS FOR $N=2$ CASE

As a prelude to understanding the nature of soliton solutions for arbitrary N-CNLS equations, we first present the bright one- and two- soliton solutions of Eq. (1) with $N=2$ (Manakov) case as given in Ref. [15] and then extend the analysis to obtain the explicit higher-order soliton solutions.

A. One-soliton solution

After restricting the power series expansion (4) as

$$g^{(j)} = \chi g_1^{(j)}, j = 1,2, \quad f = 1 + \chi^2 f_2 \tag{5}$$

and solving the resulting set of linear partial differential equations recursively, one can write down the explicit one-soliton solution as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_1^{(2)} \end{pmatrix} \frac{e^{\eta_1}}{1 + e^{\eta_1 + \eta_1^* + R}} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \frac{k_{1R} e^{i\eta_{1I}}}{\cosh\left(\eta_{1R} + \frac{R}{2} \right)}, \tag{6}$$

where

$$\eta_1 = k_1(t + ik_1 z), A_j = \frac{\alpha_1^{(j)}}{[\mu(|\alpha_1^{(1)}|^2 + |\alpha_1^{(2)}|^2)]^{1/2}}, \quad j = 1,2$$

and

$$e^R = \frac{\mu(|\alpha_1^{(1)}|^2 + |\alpha_1^{(2)}|^2)}{(k_1 + k_1^*)^2}.$$

Note that this one-soliton solution is characterized by three arbitrary complex parameters $\alpha_1^{(1)}$, $\alpha_1^{(2)}$, and k_1. Here the amplitudes of the soliton in the first and second components (modes) are given by $k_{1R}A_1$ and $k_{1R}A_2$, respectively, subject to the condition $|A_1|^2 + |A_2|^2 = 1/\mu$, while the soliton velocity in both the modes is given by $2k_{1I}$. Here k_{1R} and k_{1I} represent the real and imaginary parts of the complex parameter k_1. The quantity

$$\frac{R}{2k_{1R}} = \frac{1}{2k_{1R}} \ln\left[\frac{\mu(|\alpha_1^{(1)}|^2 + |\alpha_1^{(2)}|^2)}{(k_1 + k_1^*)^2} \right]$$

denotes the position of the soliton.

B. Two-soliton solution

The two-soliton solution of the integrable 2-CNLS system has been obtained in Ref. [15] after terminating power series (4) as

$$g^{(j)} = \chi g_1^{(j)} + \chi^3 g_3^{(j)}, \quad j = 1,2, \tag{7a}$$

$$f = 1 + \chi^2 f_2 + \chi^4 f_4, \tag{7b}$$

and again solving the resultant linear partial differential equations recursively . Then the explicit form of the two-soliton solution can be written as

$$q_j = \frac{\alpha_1^{(j)} e^{\eta_1} + \alpha_2^{(j)} e^{\eta_2} + e^{\eta_1 + \eta_1^* + \eta_2 + \delta_{1j}} + e^{\eta_1 + \eta_2 + \eta_2^* + \delta_{2j}}}{D}, \quad j = 1,2, \tag{8a}$$

where

$$D = 1 + e^{\eta_1 + \eta_1^* + R_1} + e^{\eta_1 + \eta_2^* + \delta_0} + e^{\eta_1^* + \eta_2 + \delta_0^*} + e^{\eta_2 + \eta_2^* + R_2} + e^{\eta_1 + \eta_1^* + \eta_2 + \eta_2^* + R_3}. \tag{8b}$$

In Eqs. (8), we have defined

$$\eta_i = k_i(t + ik_i z), \quad e^{\delta_0} = \frac{\kappa_{12}}{k_1 + k_2^*},$$

$$e^{R_1}=\frac{\kappa_{11}}{k_1+k_1^*},\quad e^{R_2}=\frac{\kappa_{22}}{k_2+k_2^*},$$

$$e^{\delta_{1j}}=\frac{(k_1-k_2)(\alpha_1^{(j)}\kappa_{21}-\alpha_2^{(j)}\kappa_{11})}{(k_1+k_1^*)(k_1^*+k_2)},$$

$$e^{\delta_{2j}}=\frac{(k_2-k_1)(\alpha_2^{(j)}\kappa_{12}-\alpha_1^{(j)}\kappa_{22})}{(k_2+k_2^*)(k_1+k_2^*)},$$

$$e^{R_3}=\frac{|k_1-k_2|^2}{(k_1+k_1^*)(k_2+k_2^*)|k_1+k_2^*|^2}(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21}), \tag{8c}$$

and

$$\kappa_{il}=\frac{\mu\sum_{n=1}^{2}\alpha_i^{(n)}\alpha_l^{(n)*}}{(k_i+k_l^*)},\quad i,l=1,2. \tag{8d}$$

The above most general bright two-soliton solution is characterized by six arbitrary complex parameters k_1, k_2, $\alpha_1^{(j)}$, and $\alpha_2^{(j)}$, $j=1,2$, and it corresponds to the collision of two bright solitons. Note that in Ref. [15], δ_{11}, δ_{12}, δ_{21}, and δ_{22} are called as δ_1, δ_1', δ_2, and δ_2', respectively. The redefined quantities δ_{ij}'s, $i,j=1,2$, are now used for notational simplicity.

C. Three-soliton solution

The two-soliton solution itself is very difficult to derive and complicated to analyze [15]. So obtaining the three-soliton solution is a more laborious and tedious task. However, we have successfully obtained the explicit form of the bright three-soliton solution also. In order to obtain the three-soliton solution of Eq. (1) for the $N=2$ case we terminate power series (4a) and (4b) as

$$g^{(j)}=\chi g_1^{(j)}+\chi^3 g_3^{(j)}+\chi^5 g_5^{(j)}, \tag{9a}$$

$$f=1+\chi^2 f_2+\chi^4 f_4+\chi^6 f_6,\quad j=1,2. \tag{9b}$$

Substitution of Eq. (9) into bilinear Eqs. (3a) and (3b) yields a set of linear partial differential equations at various powers of χ. The three-soliton solution consistent with these equations is

$$\begin{aligned}q_j=&\frac{\alpha_1^{(j)}e^{\eta_1}+\alpha_2^{(j)}e^{\eta_2}+\alpha_3^{(j)}e^{\eta_3}+e^{\eta_1+\eta_1^*+\eta_2+\delta_{1j}}+e^{\eta_1+\eta_1^*+\eta_3+\delta_{2j}}+e^{\eta_2+\eta_2^*+\eta_1+\delta_{3j}}}{D}\\&+\frac{e^{\eta_2+\eta_2^*+\eta_3+\delta_{4j}}+e^{\eta_3+\eta_3^*+\eta_1+\delta_{5j}}+e^{\eta_3+\eta_3^*+\eta_2+\delta_{6j}}+e^{\eta_1^*+\eta_2+\eta_3+\delta_{7j}}+e^{\eta_1+\eta_2^*+\eta_3+\delta_{8j}}}{D}\\&+\frac{e^{\eta_1+\eta_2+\eta_3^*+\delta_{9j}}+e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\eta_3+\tau_{1j}}+e^{\eta_1+\eta_1^*+\eta_3+\eta_3^*+\eta_2+\tau_{2j}}}{D}+\frac{e^{\eta_2+\eta_2^*+\eta_3+\eta_3^*+\eta_1+\tau_{3j}}}{D},\quad j=1,2,\end{aligned} \tag{10a}$$

where

$$\begin{aligned}D=&1+e^{\eta_1+\eta_1^*+R_1}+e^{\eta_2+\eta_2^*+R_2}+e^{\eta_3+\eta_3^*+R_3}+e^{\eta_1+\eta_2^*+\delta_{10}}+e^{\eta_1^*+\eta_2+\delta_{10}^*}+e^{\eta_1+\eta_3^*+\delta_{20}}+e^{\eta_1^*+\eta_3+\delta_{20}^*}+e^{\eta_2+\eta_3^*+\delta_{30}}\\&+e^{\eta_2^*+\eta_3+\delta_{30}^*}+e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_4}+e^{\eta_1+\eta_1^*+\eta_3+\eta_3^*+R_5}+e^{\eta_2+\eta_2^*+\eta_3+\eta_3^*+R_6}+e^{\eta_1+\eta_1^*+\eta_2+\eta_3^*+\tau_{10}}+e^{\eta_1+\eta_1^*+\eta_3+\eta_2^*+\tau_{10}^*}\\&+e^{\eta_2+\eta_2^*+\eta_1+\eta_3^*+\tau_{20}}+e^{\eta_2+\eta_2^*+\eta_1^*+\eta_3+\tau_{20}^*}+e^{\eta_3+\eta_3^*+\eta_1+\eta_2^*+\tau_{30}}+e^{\eta_3+\eta_3^*+\eta_1^*+\eta_2+\tau_{30}^*}+e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\eta_3+\eta_3^*+R_7}.\end{aligned} \tag{10b}$$

Here

$$\eta_i=k_i(t+ik_iz),\quad i=1,2,3, \tag{10c}$$

$$e^{\delta_{1j}}=\frac{(k_1-k_2)(\alpha_1^{(j)}\kappa_{21}-\alpha_2^{(j)}\kappa_{11})}{(k_1+k_1^*)(k_1^*+k_2)},$$

$$e^{\delta_{2j}}=\frac{(k_1-k_3)(\alpha_1^{(j)}\kappa_{31}-\alpha_3^{(j)}\kappa_{11})}{(k_1+k_1^*)(k_1^*+k_3)},$$

$$e^{\delta_{3j}}=\frac{(k_1-k_2)(\alpha_1^{(j)}\kappa_{22}-\alpha_2^{(j)}\kappa_{12})}{(k_1+k_2^*)(k_2+k_2^*)},$$

$$e^{\delta_{4j}}=\frac{(k_2-k_3)(\alpha_2^{(j)}\kappa_{32}-\alpha_3^{(j)}\kappa_{22})}{(k_2+k_2^*)(k_2^*+k_3)},$$

$$e^{\delta_{5j}}=\frac{(k_1-k_3)(\alpha_1^{(j)}\kappa_{33}-\alpha_3^{(j)}\kappa_{13})}{(k_3+k_3^*)(k_3^*+k_1)},$$

$$e^{\delta_{6j}}=\frac{(k_2-k_3)(\alpha_2^{(j)}\kappa_{33}-\alpha_3^{(j)}\kappa_{23})}{(k_3^*+k_2)(k_3^*+k_3)},$$

$$e^{\delta_{7j}}=\frac{(k_2-k_3)(\alpha_2^{(j)}\kappa_{31}-\alpha_3^{(j)}\kappa_{21})}{(k_1^*+k_2)(k_1^*+k_3)},$$

$$e^{\delta_{8j}}=\frac{(k_1-k_3)(\alpha_1^{(j)}\kappa_{32}-\alpha_3^{(j)}\kappa_{12})}{(k_1+k_2^*)(k_2^*+k_3)},$$

$$e^{\delta_{9j}}=\frac{(k_1-k_2)(\alpha_1^{(j)}\kappa_{23}-\alpha_2^{(j)}\kappa_{13})}{(k_1+k_3^*)(k_2+k_3^*)},$$

$$e^{\tau_{1j}}=\frac{(k_2-k_1)(k_3-k_1)(k_3-k_2)(k_2^*-k_1^*)}{(k_1^*+k_1)(k_1^*+k_2)(k_1^*+k_3)(k_2^*+k_1)(k_2^*+k_2)(k_2^*+k_3)}\times[\alpha_1^{(j)}(\kappa_{21}\kappa_{32}-\kappa_{22}\kappa_{31})+\alpha_2^{(j)}(\kappa_{12}\kappa_{31}-\kappa_{32}\kappa_{11})+\alpha_3^{(j)}(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21})],$$

$$e^{\tau_{2j}}=\frac{(k_2-k_1)(k_3-k_1)(k_3-k_2)(k_3^*-k_1^*)}{(k_1^*+k_1)(k_1^*+k_2)(k_1^*+k_3)(k_3^*+k_1)(k_3^*+k_2)(k_3^*+k_3)}\times[\alpha_1^{(j)}(\kappa_{33}\kappa_{21}-\kappa_{31}\kappa_{23})+\alpha_2^{(j)}(\kappa_{31}\kappa_{13}-\kappa_{11}\kappa_{33})+\alpha_3^{(j)}(\kappa_{23}\kappa_{11}-\kappa_{13}\kappa_{21})],$$

$$e^{\tau_{3j}}=\frac{(k_2-k_1)(k_3-k_1)(k_3-k_2)(k_3^*-k_2^*)}{(k_2^*+k_1)(k_2^*+k_2)(k_2^*+k_3)(k_3^*+k_1)(k_3^*+k_2)(k_3^*+k_3)}\times[\alpha_1^{(j)}(\kappa_{22}\kappa_{33}-\kappa_{23}\kappa_{32})+\alpha_2^{(j)}(\kappa_{13}\kappa_{32}-\kappa_{33}\kappa_{12})+\alpha_3^{(j)}(\kappa_{12}\kappa_{23}-\kappa_{22}\kappa_{13})], \quad (10d)$$

$$e^{R_m}=\frac{\kappa_{mm}}{k_m+k_m^*},\quad m=1,2,3,\quad e^{\delta_{10}}=\frac{\kappa_{12}}{k_1+k_2^*},$$

$$e^{\delta_{20}}=\frac{\kappa_{13}}{k_1+k_3^*},\quad e^{\delta_{30}}=\frac{\kappa_{23}}{k_2+k_3^*},$$

$$e^{R_4}=\frac{(k_2-k_1)(k_2^*-k_1^*)}{(k_1^*+k_1)(k_1^*+k_2)(k_1+k_2^*)(k_2^*+k_2)}\times[\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21}],$$

$$e^{R_5}=\frac{(k_3-k_1)(k_3^*-k_1^*)}{(k_1^*+k_1)(k_1^*+k_3)(k_3^*+k_1)(k_3^*+k_3)}\times[\kappa_{33}\kappa_{11}-\kappa_{13}\kappa_{31}],$$

$$e^{R_6}=\frac{(k_3-k_2)(k_3^*-k_2^*)}{(k_2^*+k_2)(k_2^*+k_3)(k_3^*+k_2)(k_3+k_3^*)}\times[\kappa_{22}\kappa_{33}-\kappa_{23}\kappa_{32}],$$

$$e^{\tau_{10}}=\frac{(k_2-k_1)(k_3^*-k_1^*)}{(k_1^*+k_1)(k_1^*+k_2)(k_3^*+k_1)(k_3^*+k_2)}\times[\kappa_{11}\kappa_{23}-\kappa_{21}\kappa_{13}],$$

$$e^{\tau_{20}}=\frac{(k_1-k_2)(k_3^*-k_2^*)}{(k_2^*+k_1)(k_2^*+k_2)(k_3^*+k_1)(k_3^*+k_2)}\times[\kappa_{22}\kappa_{13}-\kappa_{12}\kappa_{23}],$$

$$e^{\tau_{30}}=\frac{(k_3-k_1)(k_3^*-k_2^*)}{(k_2^*+k_1)(k_2^*+k_3)(k_3^*+k_1)(k_3^*+k_3)}\times[\kappa_{33}\kappa_{12}-\kappa_{13}\kappa_{32}],$$

$$e^{R_7}=\frac{|k_1-k_2|^2|k_2-k_3|^2|k_3-k_1|^2}{(k_1+k_1^*)(k_2+k_2^*)(k_3+k_3^*)|k_1+k_2^*|^2|k_2+k_3^*|^2|k_3+k_1^*|^2}\times[(\kappa_{11}\kappa_{22}\kappa_{33}-\kappa_{11}\kappa_{23}\kappa_{32})+(\kappa_{12}\kappa_{23}\kappa_{31}-\kappa_{12}\kappa_{21}\kappa_{33})+(\kappa_{21}\kappa_{13}\kappa_{32}-\kappa_{22}\kappa_{13}\kappa_{31})], \quad (10e)$$

and

$$\kappa_{il}=\frac{\mu\sum_{n=1}^{2}\alpha_i^{(n)}\alpha_l^{(n)*}}{(k_i+k_l^*)},\quad i,l=1,2,3. \tag{10f}$$

The above three-soliton solution represents three-soliton interaction in the 2-CNLS equations and is characterized by nine arbitrary complex parameters $\alpha_i^{(j)}$'s and k_i's, $i=1,2,3$, $j=1,2$. One can also check that the above general three-soliton solution of the 2-CNLS equations reduces to that of the solution given in Ref. [24] for the particular case of $\alpha_3^{(1)}=1$. Further, the form in which we have presented the solution eases the complexity in generalizing the solution to multicomponent case as well as to higher-order soliton solutions.

D. Four-soliton solution

The expression is quite lengthy, but it is written explicitly in terms of exponential functions so as to check the pairwise nature of collisions. We indicate the form in the Appendix. One can generalize these expressions for the arbitrary N case also. However, it is too complicated to present the explicit form and so we desist from doing so.

IV. MULTISOLITON SOLUTIONS FOR THE *N*-CNLS EQUATIONS

As mentioned in the Introduction, results are scarce for Eq. (1) with $N>2$ and there exists a large class of physical systems in which the N-CNLS equations occur naturally. Further, in the context of spatial solitons in photorefractive media, each fundamental soliton can be "spread out" into several incoherent components [25], as defined by the polarization vectors. Obtaining one-, two-, and higher-order soliton solutions of N-CNLS equations will be of considerable significance in these topics. In order to study the solution properties of such systems we consider integrable N-CNLS equations (1). Following the procedure mentioned in the preceding section we obtain the one-, two-, and three- (as well as four-) soliton solutions of N-CNLS equations as given below. Particularly the so-called partially coherent solitons will turn out to be special cases of these soliton solutions (see Sec. IX).

A. One-soliton solution

The one-soliton solution of Eq. (1) is obtained as

$$(q_1,q_2,\ldots,q_N)^T=k_{1R}e^{i\eta_{1I}}\mathrm{sech}\left(\eta_{1R}+\frac{R}{2}\right)\times(A_1,A_2,\ldots,A_N)^T, \tag{11}$$

where $\eta_1=k_1(t+ik_1z)$, $A_j=\alpha_1^{(j)}/\Delta$, $\Delta=[\mu(\Sigma_{j=1}^N|\alpha_1^{(j)}|^2)]^{1/2}$, $e^R=\Delta^2/(k_1+k_1^*)^2$, $\alpha_1^{(j)}$ and k_1, $j=1,2\ldots,N$, are $(N+1)$ arbitrary complex parameters. Further $k_{1R}A_j$ gives the amplitude of the jth mode ($j=1,2,\ldots,N$) and $2k_{1I}$ is the soliton velocity in all the N modes.

B. Two-soliton solution

The two-soliton solution of Eq. (1) can be obtained by following the procedure given for the two-component case. It can be written as

$$q_j=\frac{\alpha_1^{(j)}e^{\eta_1}+\alpha_2^{(j)}e^{\eta_2}+e^{\eta_1+\eta_1^*+\eta_2+\delta_{1j}}+e^{\eta_1+\eta_2+\eta_2^*+\delta_{2j}}}{D},\quad j=1,2,\ldots,N, \tag{12}$$

where the denominator D and the coefficients e^{R_1}, e^{R_2}, e^{R_3}, e^{δ_0}, $e^{\delta_0^*}$, $e^{\delta_{1j}}$, and $e^{\delta_{2j}}$, bear the same form as given in Eqs. (8c) and (8d), except that j now runs from 1 to N and that κ_{il}'s are redefined as

$$\kappa_{il}=\frac{\mu\sum_{n=1}^{N}\alpha_i^{(n)}\alpha_l^{(n)*}}{(k_i+k_l^*)},\quad i,l=1,2. \tag{13}$$

One may also note that the above two-soliton solution depends on $2(N+1)$ arbitrary complex parmeters $\alpha_1^{(j)}$, $\alpha_2^{(j)}$, k_1, and k_2, $j=1,2,\ldots,N$.

C. Three-soliton solution

Following the procedure given in the preceding section we obtain the three-soliton solution to the N-CNLS equations as

$$q_j=\frac{\alpha_1^{(j)}e^{\eta_1}+\alpha_2^{(j)}e^{\eta_2}+\alpha_3^{(j)}e^{\eta_3}+e^{\eta_1+\eta_1^*+\eta_2+\delta_{1j}}+e^{\eta_1+\eta_1^*+\eta_3+\delta_{2j}}+e^{\eta_2+\eta_2^*+\eta_1+\delta_{3j}}}{D}$$
$$+\frac{e^{\eta_2+\eta_2^*+\eta_3+\delta_{4j}}+e^{\eta_3+\eta_3^*+\eta_1+\delta_{5j}}+e^{\eta_3+\eta_3^*+\eta_2+\delta_{6j}}+e^{\eta_1^*+\eta_2+\eta_3+\delta_{7j}}+e^{\eta_1+\eta_2^*+\eta_3+\delta_{8j}}}{D}$$
$$+\frac{e^{\eta_1+\eta_2+\eta_3^*+\delta_{9j}}+e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\eta_3+\tau_{1j}}}{D}+\frac{e^{\eta_1+\eta_1^*+\eta_3+\eta_3^*+\eta_2+\tau_{2j}}+e^{\eta_2+\eta_2^*+\eta_3+\eta_3^*+\eta_1+\tau_{3j}}}{D},\quad j=1,2,\ldots,N. \tag{14a}$$

Here also the denominator D and all the other quantities are the same as those given under Eq. (10) except for the redefinition of κ_{il}'s as

$$\kappa_{il}=\frac{\mu\sum_{n=1}^{N}\alpha_i^{(n)}\alpha_l^{(n)*}}{(k_i+k_l^*)},\quad i,l=1,2,3. \tag{14b}$$

It can be observed from the above expression that as the number of solitons increases, the complexity also increases and the present three-soliton solution is characterized by $3(N+1)$ complex parameters $\alpha_1^{(j)}$, $\alpha_2^{(j)}$, $\alpha_3^{(j)}$, $j=1,2,\ldots,N$, k_1, k_2, and k_3.

The above procedure can be generalized to obtain the four-soliton solution and higher-order soliton solutions as discussed in the case of 2-CNLS equations straightforwardly, and one can predict that the N-soliton solution of N-CNLS will be dependent on $N(N+1)$ arbitrary complex parameters.

V. SHAPE-CHANGING NATURE OF SOLITON INTERACTIONS AND INTENSITY REDISTRIBUTIONS

The remarkable fact about the above bright soliton solutions of the integrable CNLS system is that they exhibit fascinating shape-changing (intensity redistribution or energy exchange) collisions as we will see below. This interesting behavior has been reported in Ref. [15] for the two-soliton solution of the 2-CNLS equations. In a very recent letter [20], the present authors have constructed the two-soliton solution of the 3-CNLS and generalized it to N-CNLS, for arbitrary N, and briefly indicated similar shape-changing collision dynamics of two interacting bright solitons. As these N-CNLS equations arise in diverse areas of physics as mentioned in the Introduction, it is of interest to analyze the interaction properties of the soliton solutions of 2-, 3-, and N-CNLS equations. The collision dynamics can be well understood by making an appropriate asymptotic analysis of the soliton solutions given in the previous sections. Such an analysis will then be used to identify suitable generalized linear fractional transformations in the following section, to obtain possible multistate logic.

A. Asymptotic analysis of two-soliton solution of 2-CNLS equations

To start with we shall briefly review the collision properties associated with the two-soliton solution (8) of the 2-CNLS equations discussed in Ref. [15] in order to extend the ideas of the N-CNLS case. Without loss of generality, we assume that $k_{jR}>0$ and $k_{1I}>k_{2I}$, $k_j=k_{jR}+ik_{jI}$, $j=1,2$, which corresponds to a head-on collision of the solitons (for the case $k_{1I}=k_{2I}$, see Sec. IX). For the above parametric choice, the variables η_{jR}'s (real part of η_j) for the two-solitons behave asymptotically as (i) $\eta_{1R}\sim 0$, $\eta_{2R}\to\pm\infty$ as $z\to\pm\infty$ and (ii) $\eta_{2R}\sim 0$, $\eta_{1R}\to\mp\infty$ as $z\to\pm\infty$. This leads to the following asymptotic forms for the two-soliton solution. (For other choices of k_{iR} and k_{iI}, $i=1,2$, similar analysis as given below can be performed straightforwardly.)

(i) Before collision (limit $z\to-\infty$).

(a) *Soliton 1* ($\eta_{1R}\approx 0,\eta_{2R}\to-\infty$):

$$\begin{pmatrix} q_1\\ q_2\end{pmatrix}\to\begin{pmatrix} A_1^{1-}\\ A_2^{1-}\end{pmatrix}k_{1R}e^{i\eta_{1I}}\mathrm{sech}\left(\eta_{1R}+\frac{R_1}{2}\right), \tag{15a}$$

where

$$\begin{pmatrix} A_1^{1-}\\ A_2^{1-}\end{pmatrix}=\begin{pmatrix} \alpha_1^{(1)}\\ \alpha_1^{(2)}\end{pmatrix}\frac{e^{-R_1/2}}{(k_1+k_1^*)}. \tag{15b}$$

The quantity e^{R_1} is defined in Eq. (8c).

(b) *Soliton 2* ($\eta_{2R}\approx 0,\eta_{1R}\to\infty$):

$$\begin{pmatrix} q_1\\ q_2\end{pmatrix}\to\begin{pmatrix} A_1^{2-}\\ A_2^{2-}\end{pmatrix}k_{2R}e^{i\eta_{2I}}\mathrm{sech}\left(\eta_{2R}+\frac{(R_3-R_1)}{2}\right), \tag{16a}$$

where

$$\begin{pmatrix} A_1^{2-}\\ A_2^{2-}\end{pmatrix}=\begin{pmatrix} e^{\delta_{11}}\\ e^{\delta_{12}}\end{pmatrix}\frac{e^{-(R_1+R_3)/2}}{(k_2+k_2^*)}. \tag{16b}$$

The quantities in the above expression are again defined in Eq. (8c).

(ii) After collision (limit $z\to\infty$). Similarly, for $z\to\infty$, we have the following forms for solitons S_1 and S_2.

(a) *Soliton 1* ($\eta_{1R}\approx 0,\eta_{2R}\to\infty$):

$$\begin{pmatrix} q_1\\ q_2\end{pmatrix}\to\begin{pmatrix} A_1^{1+}\\ A_2^{1+}\end{pmatrix}k_{1R}e^{i\eta_{1I}}\mathrm{sech}\left(\eta_{1R}+\frac{(R_3-R_2)}{2}\right), \tag{17a}$$

where

$$\begin{pmatrix} A_1^{1+}\\ A_2^{1+}\end{pmatrix}=\begin{pmatrix} e^{\delta_{21}}\\ e^{\delta_{22}}\end{pmatrix}\frac{e^{-(R_2+R_3)/2}}{(k_1+k_1^*)}. \tag{17b}$$

(b) *Soliton 2* ($\eta_{2R}\approx 0,\eta_{1R}\to-\infty$):

$$\begin{pmatrix} q_1\\ q_2\end{pmatrix}\to\begin{pmatrix} A_1^{2+}\\ A_2^{2+}\end{pmatrix}k_{2R}e^{i\eta_{2I}}\mathrm{sech}\left(\eta_{2R}+\frac{R_2}{2}\right), \tag{18a}$$

where

$$\begin{pmatrix} A_1^{2+}\\ A_2^{2+}\end{pmatrix}=\begin{pmatrix} \alpha_2^{(1)}\\ \alpha_2^{(2)}\end{pmatrix}\frac{e^{-R_2/2}}{(k_2+k_2^*)}. \tag{18b}$$

In the above expressions for S_1 and S_2 after collision the quantities e^{R_2},e^{R_3}, $e^{\delta_{21}}$, and $e^{\delta_{22}}$ are defined in Eq. (8c).

B. Asymptotic analysis of the two-soliton solution of N-CNLS equations

We require the asymptotic forms of the two-soliton solutions for arbitrary N case in the following section in order to identify a generalized linear fractional transformation for the amplitude redistribution among the components. To get the asymptotic forms of two-soliton solution of the N-CNLS case, as may be checked by a careful asymptotic analysis along the lines of the $N=2$ case, we simply increase the number of components in the $A^{\pm}$ vectors above up to $N[A^{\pm}=(A_1^{\pm},A_2^{\pm},\ldots,A_N^{\pm})^T]$ by adding two more complex parameters $\alpha_1^{(i)}$, $\alpha_2^{(i)}, i=3,4,\ldots,N$, to each of the components so that the forms of the quantities $e^{R_1},e^{R_2},e^{R_3},e^{\delta_{11}}$, $e^{\delta_{12}},e^{\delta_{21}},e^{\delta_{22}}$ in Eq. (8c) remain the same as above except for the replacement of the range of the summation in κ_{il} [Eq. (8d)] from $n=1,2$ to $n=1,2,\ldots,N$. As an example, in the following we give the asymptotic forms of two-soliton solution of the N-CNLS equations with $N=3$, for the case $k_{lR}>0, l=1,2$, and $k_{1I}>k_{2I}$. For other possibilities similar analysis can be made.

(i) Before collision (limit $z\rightarrow-\infty$).

(a) *Soliton 1* ($\eta_{1R}\approx 0, \eta_{2R}\rightarrow-\infty$):

$$\begin{pmatrix} q_1\\ q_2\\ q_3\end{pmatrix}\approx\begin{pmatrix} A_1^{1-}\\ A_2^{1-}\\ A_3^{1-}\end{pmatrix} k_{1R}e^{i\eta_{1I}}\operatorname{sech}\left(\eta_{1R}+\frac{R_1}{2}\right), \tag{19a}$$

where

$$\begin{pmatrix} A_1^{1-}\\ A_2^{1-}\\ A_3^{1-}\end{pmatrix}=\begin{pmatrix} \alpha_1^{(1)}\\ \alpha_1^{(2)}\\ \alpha_1^{(3)}\end{pmatrix}\frac{e^{-R_1/2}}{(k_1+k_1^*)}. \tag{19b}$$

(b) *Soliton 2* ($\eta_{2R}\approx 0, \eta_{1R}\rightarrow\infty$):

$$\begin{pmatrix} q_1\\ q_2\\ q_3\end{pmatrix}\approx\begin{pmatrix} A_1^{2-}\\ A_2^{2-}\\ A_3^{2-}\end{pmatrix} k_{2R}e^{i\eta_{2I}}\operatorname{sech}\left(\eta_{2R}+\frac{(R_3-R_1)}{2}\right), \tag{20a}$$

where

$$\begin{pmatrix} A_1^{2-}\\ A_2^{2-}\\ A_3^{2-}\end{pmatrix}=\begin{pmatrix} e^{\delta_{11}}\\ e^{\delta_{12}}\\ e^{\delta_{13}}\end{pmatrix}\frac{e^{-(R_1+R_3)/2}}{(k_2+k_2^*)}. \tag{20b}$$

(ii) After collision (limit $z\rightarrow\infty$).

(a) *Soliton 1* ($\eta_{1R}\approx 0, \eta_{2R}\rightarrow\infty$):

$$\begin{pmatrix} q_1\\ q_2\\ q_3\end{pmatrix}\approx\begin{pmatrix} A_1^{1+}\\ A_2^{1+}\\ A_3^{1+}\end{pmatrix} k_{1R}e^{i\eta_{1I}}\operatorname{sech}\left(\eta_{1R}+\frac{(R_3-R_2)}{2}\right), \tag{21a}$$

where

$$\begin{pmatrix} A_1^{1+}\\ A_2^{1+}\\ A_3^{1+}\end{pmatrix}=\begin{pmatrix} e^{\delta_{21}}\\ e^{\delta_{22}}\\ e^{\delta_{23}}\end{pmatrix}\frac{e^{-(R_2+R_3)/2}}{(k_1+k_1^*)}. \tag{21b}$$

(b) *Soliton 2* ($\eta_{2R}\approx 0, \eta_{1R}\rightarrow-\infty$):

$$\begin{pmatrix} q_1\\ q_2\\ q_3\end{pmatrix}\approx\begin{pmatrix} A_1^{2+}\\ A_2^{2+}\\ A_3^{2+}\end{pmatrix} k_{2R}e^{i\eta_{2I}}\operatorname{sech}\left(\eta_{2R}+\frac{R_2}{2}\right), \tag{22a}$$

where

$$\begin{pmatrix} A_1^{2+}\\ A_2^{2+}\\ A_3^{2+}\end{pmatrix}=\begin{pmatrix} \alpha_2^{(1)}\\ \alpha_2^{(2)}\\ \alpha_2^{(3)}\end{pmatrix}\frac{e^{-R_2/2}}{(k_2+k_2^*)}. \tag{22b}$$

In the above expressions, the forms of the quantities e^{R_j}, $e^{\delta_{ij}}, i=1,2$, $j=1,2,3$, can be identified from Eqs. (12) and (13) with $N=3$.

1. Intensity redistribution

The above analysis clearly shows that due to the interaction between two copropagating solitons S_1 and S_2 in an N-CNLS system, their amplitudes change from $A_j^{1-}k_{1R}$ and $A_j^{2-}k_{2R}$ to $A_j^{1+}k_{1R}$ and $A_j^{2+}k_{2R}$, $j=1,2,\ldots,N$, respectively. However, during the interaction process the total energy of each of the solitons is conserved, that is,

$$\sum_{j=1}^{N}|A_j^{1\pm}|^2=\sum_{j=1}^{N}|A_j^{2\pm}|^2=\frac{1}{\mu}. \tag{23}$$

Note that this is a consequence of the conservation of L^2 norm. Another noticeable observation of this interaction process is that one can observe from the equation of motion (1) itself, that the intensity of each of the modes is separately conserved, that is,

$$\int_{-\infty}^{\infty}|q_j|^2dz=\text{const},\quad j=1,2,\ldots,N. \tag{24}$$

The above two equations (23) and (24) ensure that in a two-soliton collision process (as well as in multisoliton collision processes as will be seen later on), the total intensity of individual solitons in all the N modes are conserved along with conservation of intensity of individual modes (even while allowing an intensity redistribution). This is a striking feature of the integrable nature of multicomponent CNLS equations (1). The change in the amplitude of each of the solitons in the jth mode can be obtained by introducing the transition matrix T_j^l, $j=1,2,\ldots,N$, $l=1,2$, such that

$$A_j^{l+}=T_j^lA_j^{l-}. \tag{25a}$$

The form of T_j^l's can be obtained from the above asymptotic analysis as

$$T_j^1=\left(\frac{a_2}{a_2^*}\right)\sqrt{\frac{\kappa_{21}}{\kappa_{12}}}\left[\frac{1-\lambda_2\left(\dfrac{\alpha_2^{(j)}}{\alpha_1^{(j)}}\right)}{\sqrt{1-\lambda_1\lambda_2}}\right],\quad j=1,2,\ldots,N, \tag{25b}$$

where

TABLE I. Possible combinations of intensity redistribution among the modes of soliton S_1 in the two-soliton collision process.

(a) $N=2$ case

Case	q_1	q_2
1	E	S
2	S	E

(b) $N=3$ case

Case	q_1	q_2	q_3
1	E	S	S
2	S	E	S
3	S	S	E
4	S	E	E
5	E	S	E
6	E	E	S

$$a_2=(k_2+k_1^*)\left[(k_1-k_2)\sum_{n=1}^{N}\alpha_1^{(n)}\alpha_2^{(n)*}\right]^{1/2}, \tag{25c}$$

and

$$T_j^2=-\left(\frac{a_1}{a_1^*}\right)\sqrt{\frac{\kappa_{21}}{\kappa_{12}}}\left[\frac{\sqrt{1-\lambda_1\lambda_2}}{1-\lambda_1\left(\frac{\alpha_1^{(j)}}{\alpha_2^{(j)}}\right)}\right], \quad j=1,2,\ldots,N, \tag{25d}$$

in which

$$a_1=(k_1+k_2^*)\left[(k_1-k_2)\sum_{n=1}^{N}\alpha_1^{(n)*}\alpha_2^{(n)}\right]^{1/2}. \tag{25e}$$

In the above expressions $\lambda_1=\kappa_{21}/\kappa_{11}$ and $\lambda_2=\kappa_{12}/\kappa_{22}$, where κ_{il}'s, $i,l=1,2$, are defined in Eq. (13). Then the intensity exchange in solitons S_1 and S_2 due to collision can be obtained by taking the absolute square of Eqs. (25b) and (25d), respectively.

The above expressions for the components of the transition matrix implies that in general there is a redistribution of the intensities in the N modes of both the solitons after collision. Only for the special case

$$\frac{\alpha_1^{(1)}}{\alpha_2^{(1)}}=\frac{\alpha_1^{(2)}}{\alpha_2^{(2)}}=\cdots=\frac{\alpha_1^{(N)}}{\alpha_2^{(N)}} \tag{26}$$

does the standard elastic collision occur. For all other choices of the parameters, shape-changing (intensity redistribution) collision occurs.

The two conservation relations (23) and (24) allow the intensity redistribution to take place in definite ways. In general, for N-CNLS equations the intensity redistribution in a two-soliton collision can occur in 2^N-2 ways. Denoting E and S as enhancement and suppression, respectively, either complete or partial, of the intensity of corresponding modes, we table below (Table I) the possibilities of intensity redistribution for the case $N=2$ and $N=3$.

For each of the above choices of S_1, the form of S_2 is determined by the conserved quantity (24) for the intensities of the individual modes. For illustrative purposes, we have shown in Figs. 1 and 2 a few of such possibilities of intensity switching for the $N=2$ and $N=3$ cases, respectively.

2. Phase shifts

Further, from the asymptotic forms of the solitons S_1 and S_2, it can be observed that the phases of solitons S_1 and S_2 also change during a collision process and that the phase shifts are now not only functions of the parameters k_1 and k_2 but also dependent on $\alpha_i^{(j)}$'s, $i=1,2$, $j=1,2,\ldots,N$. The phase shift suffered by the soliton S_1 during collision is

$$\Phi^1=\frac{(R_3-R_1-R_2)}{2}=\left(\frac{1}{2}\right)\ln\left[\frac{|k_1-k_2|^2(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21})}{|k_1+k_2^*|^2\kappa_{11}\kappa_{22}}\right], \tag{27}$$

where κ_{il}'s are defined in Eq. (13). Similarly the soliton S_2 suffers a phase shift

$$\Phi^2=-\frac{(R_3-R_2-R_1)}{2}=-\Phi^1. \tag{28}$$

Then the absolute value of phase shift suffered by the two-solitons is

$$|\Phi|=|\Phi^1|=|\Phi^2|. \tag{29}$$

Let us consider the case $N=2$. For a better understanding, let us consider the pure elastic collision case $(\alpha_1^{(1)}:\alpha_2^{(1)}=\alpha_1^{(2)}:\alpha_2^{(2)})$ corresponding to parallel modes. Here the absolute phase shift [see Eq. (29)] can be obtained as

$$|\Phi|=\left|\ln\left[\frac{|k_1-k_2|^2}{|k_1+k_2^*|^2}\right]\right|=2\left|\ln\left[\frac{|k_1-k_2|}{|k_1+k_2^*|}\right]\right|. \tag{30}$$

Similarly for the case corresponding to orthogonal modes $(\alpha_1^{(1)}:\alpha_2^{(1)}=\infty,\alpha_1^{(2)}:\alpha_2^{(2)}=0)$ the absolute phase shift is found from Eqs. (27)–(29) to be

$$|\Phi|=\left|\ln\left[\frac{|k_1-k_2|}{|k_1+k_2^*|}\right]\right|. \tag{31}$$

The absolute value of the phase shift takes intermediate values for other choices of the parameters $\alpha_i^{(j)}$'s, $i=1,2$, $j=1,2,\ldots,N$. Thus phase shifts do vary depending on $\alpha_i^{(j)}$'s (amplitudes) for fixed k_i's. In Fig. 3, we plot the change of $|\Phi|$ as a function of $\alpha_1^{(1)}$, when it is real, at $\alpha_1^{(2)}=\alpha_2^{(2)}=1$, $\alpha_2^{(1)}=(39+80i)/89$, $k_1=1+i$, and $k_2=2-i$. Similar analysis can be done for the $N=3$ case and for the arbitrary N case.

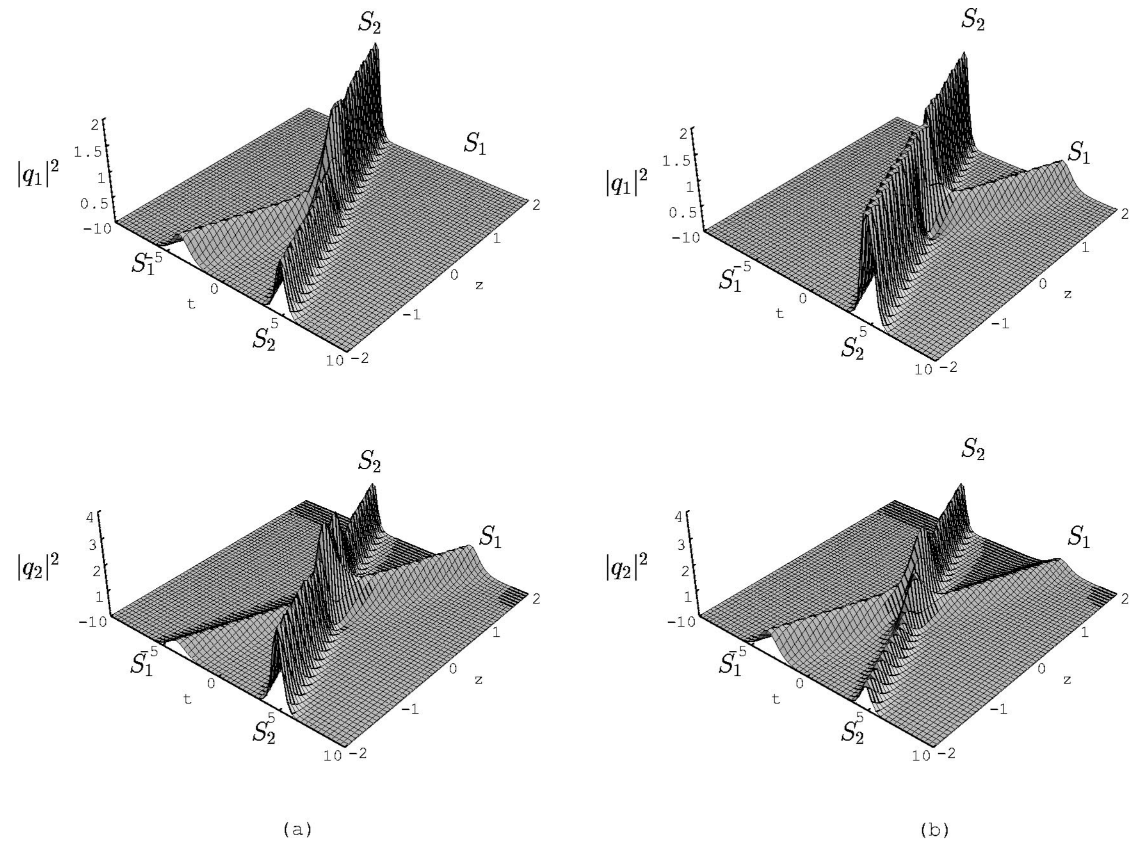


FIG. 1. Two distinct possibilities of the shape-changing two-soliton collision in the integrable 2-CNLS system. The parameters are chosen as (a) $k_1=1+i,\ k_2=2-i,\ \alpha_1^{(1)}=\alpha_1^{(2)}=\alpha_2^{(2)}=1,\ \alpha_2^{(1)}=(39+80i)/89$; (b) $k_1=1+i,\ k_2=2-i,\ \alpha_1^{(1)}=0.02+0.1i,\ \alpha_1^{(2)}=\alpha_2^{(1)}=\alpha_2^{(2)}=1$.

3. Relative separation distance

Ultimately, the above phase shifts make the relative separation distance $t_{12}^{\pm}$ between the solitons [that is, the position of S_2 (at $z\rightarrow\pm\infty$) minus position of S_1 (at $z\rightarrow\pm\infty$)] also to vary during collision, depending upon the amplitudes of the modes. The change in the relative separation distance is found to be

$$\Delta t_{12}=t_{12}^{-}-t_{12}^{+}=\frac{(k_{1R}+k_{2R})}{k_{1R}k_{2R}}\Phi^1. \tag{32}$$

Thus as a whole the intensity profiles of the two-solitons in different modes as well as the phases, and hence the relative separation distance are nontrivially dependent on $\alpha_i^{(j)}$'s and vary as a result of soliton interaction.

VI. GENERALIZED LINEAR FRACTIONAL TRANSFORMATIONS AND MULTISTATE LOGIC

The intensity redistribution was characterized by the transition matrix as given in Eq. (25) in the preceding section. Interestingly, this redistribution can also be viewed as a linear fractional transformation (LFT) as already pointed out by Jakubowski *et al.* [18]. However, no systematic derivation of such a connection was made. In this section, we point out that in fact a reformulation of Eq. (25) allows one to deduce such an LFT in a systematic way. This in turn allows us to generalize the procedure to the N-component case leading to a generalized LFT for the amplitude change during soliton collision thereby leading to a multistate logic.

A. $N=2$ case

For the $N=2$ case, the amplitude change in the two modes of soliton 1 after interaction given by Eq. (25) can be reexpressed by the following transformation, which can be deduced from comparison of expressions (15b) and (17b):

$$A_1^{1+}=\Gamma C_{11}A_1^{1-}+\Gamma C_{12}A_2^{1-},$$

$$A_2^{1+}=\Gamma C_{21}A_1^{1-}+\Gamma C_{22}A_2^{1-}. \tag{33a}$$

Here

$$\Gamma=\Gamma(A_1^{1-},A_2^{1-},A_1^{2-},A_2^{2-})$$
$$\equiv\left(\frac{a_2}{a_2^*}\right)\left[\frac{1}{(\alpha_1^{(1)}\alpha_2^{(1)*}+\alpha_1^{(2)}\alpha_2^{(2)*})(\alpha_2^{(1)}\alpha_2^{(1)*}+\alpha_2^{(2)}\alpha_2^{(2)*})}\right]$$
$$\times\left[\frac{1}{|\kappa_{12}|^2}-\frac{1}{\kappa_{11}\kappa_{22}}\right]^{-1/2}, \tag{33b}$$

in which a_2 is given in Eq. (25c). The forms of C_{ij}'s, $i,j=1,2$, read as

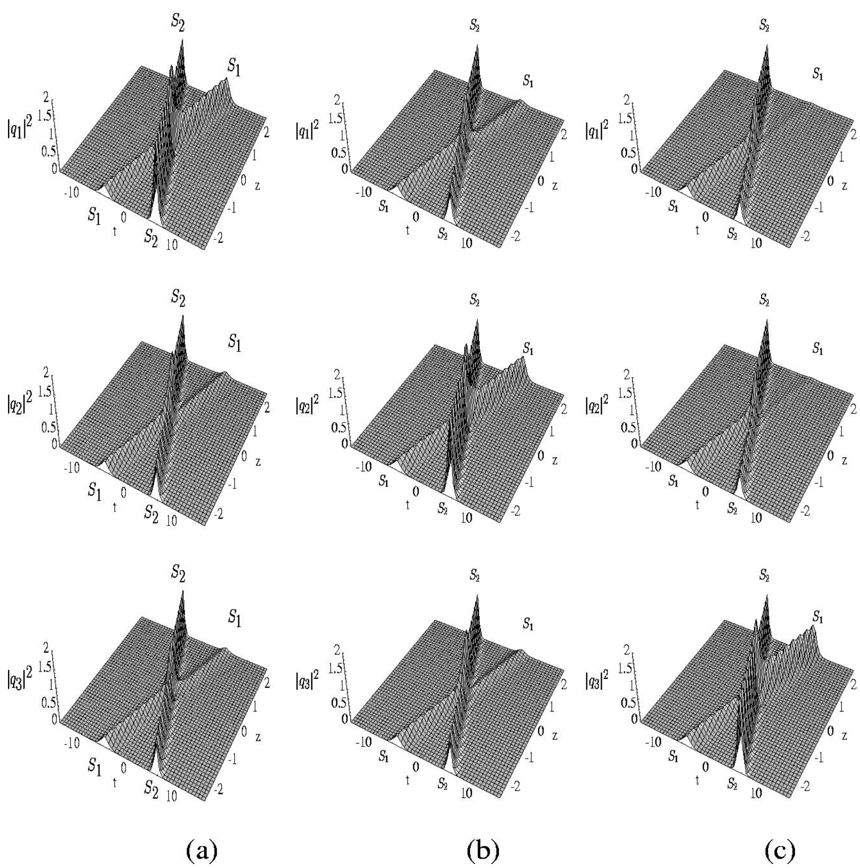


FIG. 2. Intensity profiles of the three modes of the two-soliton solution in a waveguide described by the 3-CNLS [Eq. (1) with $N=3$] showing different dramatic scenarios of the shape-changing collision for various choices of parameters.

$$C_{11}=\alpha_2^{(1)}\alpha_2^{(1)*}(k_1-k_2)+\alpha_2^{(2)}\alpha_2^{(2)*}(k_1+k_2^*), \tag{33c}$$

$$C_{12}=-\alpha_2^{(1)}\alpha_2^{(2)*}(k_2+k_2^*), \tag{33d}$$

$$C_{21}=-\alpha_2^{(2)}\alpha_2^{(1)*}(k_2+k_2^*), \tag{33e}$$

$$C_{22}=\alpha_2^{(1)}\alpha_2^{(1)*}(k_1+k_2^*)+\alpha_2^{(2)}\alpha_2^{(2)*}(k_1-k_2). \tag{33f}$$

Note that the coefficients C_{ij}'s are independent of $\alpha_1^{(j)}$'s and so of A_1^{1-} and A_2^{1-}, that is the α parameters of soliton 1. Then from Eqs. (33a) the ratios of the $A_i^{j\pm}$'s, $i,j=1,2$, can be connected through an LFT. For example, for soliton 1, from Eq. (33a),

$$\rho_{1,2}^{1+}=\frac{A_1^{1+}}{A_2^{1+}}=\frac{C_{11}\rho_{1,2}^{1-}+C_{12}}{C_{21}\rho_{1,2}^{1-}+C_{22}}, \tag{34}$$

where $\rho_{1,2}^{1-}=A_1^{1-}/A_2^{1-}$, in which the superscripts represent the underlying soliton and the subscripts represent the corresponding modes. The quantities $\rho_{1,2}^{1+}$, $\rho_{1,2}^{1-}, C_{11}, C_{12}, C_{21}, C_{22}$, in Eq. (34) are same as the quantities ρ_R, ρ_1, $[(1-h^*)/\rho_L^*+\rho_L]$, $h^*\rho_L/\rho_L^*, h^*$ and $[(1-h^*)\rho_L+1/\rho_L^*]$, respectively, given by Eq. (9) in Ref. [18] in an *adhoc* way. Thus the state of S_1 before and after interaction is characterized by $\rho_{1,2}^{1-}$ and $\rho_{1,2}^{1+}$, respectively. It is to be noticed that during collision k_i's, $i=1,2$, are unaltered. The LFT has been profitably used in Ref. [19] to construct logic gates, associated with the binary logic $\rho=[0,1]$. Similar analysis can be done for the soliton 2 also.

B. $N=3$ case

Extending the above analysis, straightforwardly one can relate the $A_j^{1\pm}$'s, $j=1,2,3$, for soliton 1, from Eqs. (19b) and (21b), as

$$A_1^{1+}=\Gamma C_{11}A_1^{1-}+\Gamma C_{12}A_2^{1-}+\Gamma C_{13}A_3^{1-}, \tag{35a}$$

$$A_2^{1+}=\Gamma C_{21}A_1^{1-}+\Gamma C_{22}A_2^{1-}+\Gamma C_{23}A_3^{1-}, \tag{35b}$$

$$A_3^{1+}=\Gamma C_{31}A_1^{1-}+\Gamma C_{32}A_2^{1-}+\Gamma C_{33}A_3^{1-}, \tag{35c}$$

where

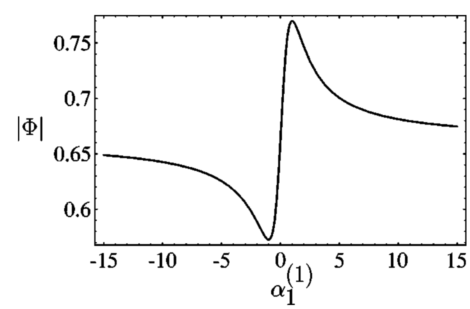


FIG. 3. Plot of the magnitude of phase shift as a function of the parameter $\alpha_1^{(1)}$, when it is real (for illustrative purposes); see Eqs. (29)–(31). The other parameters are chosen as $k_1=1+i$, $k_2=2-i$, $\alpha_1^{(2)}=\alpha_2^{(2)}=1$, and $\alpha_2^{(1)}=(39+80i)/89$.

$$\Gamma = \Gamma(A_1^{1-},A_2^{1-},A_3^{1-},A_1^{2-},A_2^{2-},A_3^{2-}) \equiv \left(\frac{a_2}{a_2^*}\right)\left[\frac{1}{(\alpha_1^{(1)}\alpha_2^{(1)*}+\alpha_1^{(2)}\alpha_2^{(2)*}+\alpha_1^{(3)}\alpha_2^{(3)*})(\alpha_2^{(1)}\alpha_2^{(1)*}+\alpha_2^{(2)}\alpha_2^{(2)*}+\alpha_2^{(3)}\alpha_2^{(3)*})}\right]\left[\frac{1}{|\kappa_{12}|^2}-\frac{1}{\kappa_{11}\kappa_{22}}\right]^{-1/2}, \tag{35d}$$

in which a_2's are redefined as

$$a_2=(k_2+k_1^*)[(k_1-k_2)(\alpha_1^{(1)}\alpha_2^{(1)*}+\alpha_1^{(2)}\alpha_2^{(2)*}+\alpha_1^{(3)}\alpha_2^{(3)*})]^{1/2}, \tag{35e}$$

and κ_{il}'s can be written from Eq. (13) with $N=3$. Note that the form of Γ is a straightforward extension of the $N=2$ case. In the above equations the coefficients C_{ij}'s, $i,j=1,2,3$, for the 3-CNLS case can be written down straightforwardly by generalizing expressions (33) corresponding to the two-soliton solution of the two-component case.

Thus in the two-soliton collision process of the $N=3$ case, for soliton 1 we obtain the generalized Möbius transformation,

$$\rho_{1,3}^{1+}=\frac{A_1^{1+}}{A_3^{1+}}=\frac{C_{11}\rho_{1,3}^{1-}+C_{12}\rho_{2,3}^{1-}+C_{13}}{C_{31}\rho_{1,3}^{1-}+C_{32}\rho_{2,3}^{1-}+C_{33}}, \tag{36a}$$

$$\rho_{2,3}^{1+}=\frac{A_2^{1+}}{A_3^{1+}}=\frac{C_{21}\rho_{1,3}^{1-}+C_{22}\rho_{2,3}^{1-}+C_{23}}{C_{31}\rho_{1,3}^{1-}+C_{32}\rho_{2,3}^{1-}+C_{33}}, \tag{36b}$$

where $\rho_{1,3}^{1-}=A_1^{1-}/A_3^{1-}$ and $\rho_{2,3}^{1-}=A_2^{1-}/A_3^{1-}$. Similar relations can be obtained for the soliton 2 also.

C. Arbitrary *N* case

Proceeding in a similar fashion one can construct for the soliton S_1 a generalized linear fractional transformation for the *N*-component case also which relates the ρ vectors before and after collision,

$$\rho_{i,N}^{1+}=\frac{A_i^{1+}}{A_N^{1+}}=\frac{\sum_{j=1}^{N}C_{ij}\rho_{j,N}^{1-}}{\sum_{j=1}^{N}C_{Nj}\rho_{j,N}^{1-}}, \tag{37a}$$

with the condition

$$\rho_{NN}^{1-}=1. \tag{37b}$$

Here $\rho_{i,N}^{1-}=A_i^{1-}/A_N^{1-}$. Similar expression can be obtained for soliton 2 also.

The above generalization paves the way not only for writing down the bilinear transformation but also for identifying multistate logic. For example, in the $N=3$ case, the following states are possible: $\rho=[\rho_1,\rho_2]\equiv[(0,0),(0,1),(1,0),(1,1)]$, where the logical "0" state can stand for the complex valued ρ state corresponding to a suppression of the amplitude in that mode, while the logical "1" state may correspond to enhancement (including no change), which can be used to perform logical operations, whereas in the $N=2$ case we have only the two state logic, $\rho=[0,1]$. This shows that for $N>2$, we will get multistate logic and we believe that such states can be of a distinct advantage in computation. This kind of study is in progress.

VII. HIGHER-ORDER SOLITON SOLUTIONS AND THEIR INTERACTIONS

Now it is of interest to study the nature of multisoliton collisions making use of the explicit forms of multisoliton solutions given in Secs. III and IV. Due to the complicated nature of the above bright soliton expressions, it becomes nontrivial to identify the nature of the collision process. In his paper [13], Manakov pointed out that in general an *N*-soliton collision does not reduce to a pair collision due to the nontrivial dependence of the amplitude of a particular soliton before interaction on the other soliton parameters. In this section by a careful asymptotic analysis of the three-soliton solution (10) of the 2-CNLS equations, which can be deduced to the *N*-CNLS case without any difficulty, we explicitly demonstrate that the collision process indeed can be considered to occur pairwise, thereby putting Manakov's statement into proper perspective and making it clearer. One can carry out a similar analysis for the four-soliton solution given in the Appendix, generalizing which one can show that in the higher-order solitons of CNLS equations also the collision is pairwise. Such an analysis also reveals the many possibilities for energy exchange among the modes of the solitons, including the exciting possibility of state restoration in higher-order soliton solutions, a precursor to the construction of logic gates.

A. Asymptotic analysis of three-soliton solution of 2-CNLS equations

Considering the explicit three-soliton expression (10), without loss of generality, we assume that the quantities k_{1R}, k_{2R}, and k_{3R} are positive and $k_{1I}>k_{2I}>k_{3I}$ (for the equal sign cases $k_{1I}=k_{2I}=k_{3I}$, see Sec. IX below). One can carry out a similar analysis for other possibilities of k_{iI}'s, $i=1,2,3$, also as discussed below. Then for the above condition the variables η_{iR}'s, $i=1,2,3$, for the three-solitons (S_1,S_2, and S_3) take the following values asymptotically:

(i) $\eta_{1R}\approx 0, \eta_{2R}\to\pm\infty, \quad \eta_{3R}\to\pm\infty, \quad$ as $\quad z\to\pm\infty$,

(ii) $\eta_{2R}\approx 0, \quad \eta_{1R}\to\mp\infty, \quad \eta_{3R}\to\pm\infty,$ as $z\to\pm\infty$,

(iii) $\eta_{3R}\approx 0, \eta_{1R}\to\mp\infty, \quad \eta_{2R}\to\mp\infty, \quad$ as $\quad z\to\pm\infty$.

Defining the various quantities R_i's, $i=1,2,\ldots,7$, δ_{lj}'s, $l=1,2,\ldots,9,j=1,2,\tau_{mj}$'s, and τ_{m0}'s, $m=1,2,3$, as in Eq. (10) we have the following limiting forms of the three-soliton solution, Eq. (10).

(i) Before collision (limit $z\to-\infty$).

(a) *Soliton 1* ($\eta_{1R}\approx 0,\eta_{2R}\to-\infty,\eta_{3R}\to-\infty$):

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \approx \begin{pmatrix} A_1^{1-} \\ A_2^{1-} \end{pmatrix} k_{1R}\mathrm{sech}\left(\eta_{1R}+\frac{R_1}{2}\right)e^{i\eta_{1I}}, \tag{38a}$$

$$\begin{pmatrix} A_1^{1-} \\ A_2^{1-} \end{pmatrix} = \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_1^{(2)} \end{pmatrix} \frac{e^{c-R_1/2}}{(k_1+k_1^*)}. \tag{38b}$$

(b) *Soliton 2* ($\eta_{2R}\approx 0,\eta_{1R}\to\infty,\eta_{3R}\to-\infty$):

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \approx \begin{pmatrix} A_1^{2-} \\ A_2^{2-} \end{pmatrix} k_{2R}\mathrm{sech}\left(\eta_{2R}+\frac{R_4-R_1}{2}\right)e^{i\eta_{2I}}, \tag{39a}$$

$$\begin{pmatrix} A_1^{2-} \\ A_2^{2-} \end{pmatrix} = \begin{pmatrix} e^{\delta_{11}} \\ e^{\delta_{12}} \end{pmatrix} \frac{e^{-(R_1+R_4)/2}}{(k_2+k_2^*)}. \tag{39b}$$

(c) *Soliton 3* ($\eta_{3R}\approx 0,\eta_{1R}\to\infty,\eta_{2R}\to\infty$):

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \approx \begin{pmatrix} A_1^{3-} \\ A_2^{3-} \end{pmatrix} k_{3R}\mathrm{sech}\left(\eta_{3R}+\frac{R_7-R_4}{2}\right)e^{i\eta_{3I}}, \tag{40a}$$

$$\begin{pmatrix} A_1^{3-} \\ A_2^{3-} \end{pmatrix} = \begin{pmatrix} e^{\tau_{11}} \\ e^{\tau_{12}} \end{pmatrix} \frac{e^{-(R_4+R_7)/2}}{(k_3+k_3^*)}. \tag{40b}$$

(ii) After collision (limit $z\to+\infty$).

(a) *Soliton 1* ($\eta_{1R}\approx 0,\eta_{2R}\to\infty,\eta_{3R}\to\infty$):

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \approx \begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix} k_{1R}\mathrm{sech}\left(\eta_{1R}+\frac{R_7-R_6}{2}\right)e^{i\eta_{1I}}, \tag{41a}$$

$$\begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix} = \begin{pmatrix} e^{\tau_{31}} \\ e^{\tau_{32}} \end{pmatrix} \frac{e^{-(R_6+R_7)/2}}{(k_1+k_1^*)}. \tag{41b}$$

(b) *Soliton 2* ($\eta_{2R}\approx 0,\eta_{1R}\to-\infty,\eta_{3R}\to\infty$):

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \approx \begin{pmatrix} A_1^{2+} \\ A_2^{2+} \end{pmatrix} k_{2R}\mathrm{sech}\left(\eta_{2R}+\frac{R_6-R_3}{2}\right)e^{i\eta_{2I}}, \tag{42a}$$

$$\begin{pmatrix} A_1^{2+} \\ A_2^{2+} \end{pmatrix} = \begin{pmatrix} e^{\delta_{61}} \\ e^{\delta_{62}} \end{pmatrix} \frac{e^{-(R_3+R_6)/2}}{(k_2+k_2^*)}. \tag{42b}$$

(c) *Soliton 3* ($\eta_{3R}\approx 0,\eta_{1R}\to-\infty,\eta_{2R}\to-\infty$):

$$\begin{pmatrix} q_1^{3+} \\ q_2^{3+} \end{pmatrix} \approx \begin{pmatrix} A_1^{3+} \\ A_2^{3+} \end{pmatrix} k_{3R}\mathrm{sech}\left(\eta_{3R}+\frac{R_3}{2}\right)e^{i\eta_{3I}}, \tag{43a}$$

$$\begin{pmatrix} A_1^{3+} \\ A_2^{3+} \end{pmatrix} = \begin{pmatrix} \alpha_3^{(1)} \\ \alpha_3^{(2)} \end{pmatrix} \frac{e^{-R_3/2}}{(k_3+k_3^*)}. \tag{43b}$$

B. Transition elements

The above analysis clearly shows that during the three-soliton interaction process, there is a redistribution of intensities among these solitons in the two modes along with amplitude dependent phase shifts as in the case of the two-soliton interaction. The amplitude changes can be expressed in terms of a transition matrix T_j^l as

$$A_j^{l+}=T_j^l A_j^{l-}, \quad j=1,2, \quad l=1,2,3. \tag{44}$$

Explicit forms of the entries of the transition matrix quantifying the amount of intensity redistribution for the three-solitons are as follows.

Soliton 1:

$$\begin{pmatrix} T_1^1 \\ T_2^1 \end{pmatrix} = \begin{pmatrix} \dfrac{e^{\tau_{31}}}{\alpha_1^{(1)}} \\ \dfrac{e^{\tau_{32}}}{\alpha_1^{(2)}} \end{pmatrix} e^{-(R_6+R_7-R_1)/2}. \tag{45a}$$

Soliton 2:

$$\begin{pmatrix} T_1^2 \\ T_2^2 \end{pmatrix} = \begin{pmatrix} e^{\delta_{61}-\delta_{11}} \\ e^{\delta_{62}-\delta_{12}} \end{pmatrix} e^{-(R_3+R_6-R_1-R_4)/2}. \tag{45b}$$

Soliton 3:

$$\begin{pmatrix} T_1^3 \\ T_2^3 \end{pmatrix} = \begin{pmatrix} \alpha_3^{(1)}e^{-\tau_{11}} \\ \alpha_3^{(2)}e^{-\tau_{12}} \end{pmatrix} e^{-(R_3-R_4-R_7)/2}. \tag{45c}$$

The various quantities found in the above equations are defined in Eq. (10).

C. Phase shifts

Now let us look into the phase shifts suffered by each of the solitons during collision. These can be written as

$$S_1:\Phi^1=\frac{R_7-R_6-R_1}{2}, \tag{46a}$$

$$S_2:\Phi^2=\frac{R_6-R_3-R_4+R_1}{2}, \tag{46b}$$

$$S_3:\Phi^3=\frac{R_3-R_7+R_4}{2}. \tag{46c}$$

Here the quantities $R_1,R_2,\ldots,R_7$ are as given in Eq. (10). Note that each of the phase shifts Φ^1,Φ^2, and Φ^3 contains a part which depends purely on k_i's, $i=1,2,3$, and another part which depends on the amplitude (polarization) parameters $\alpha_i^{(j)}$'s along with k_i's.

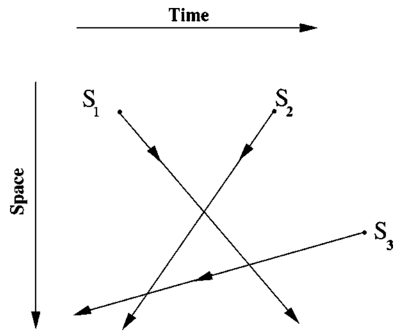


FIG. 4. A schematic three-soliton collision process (for the choice $k_{1R},k_{2R},k_{3R}>0, k_{1I}>k_{2I}>k_{3I}$). The effects of phase shifts are not included in the figure.

D. Relative separation distances

As a consequence of the above amplitude dependent phase shifts, the relative separation distances between the solitons $t_{ij}^{\pm}$ [position of S_j (at $z\to\pm\infty$) — position of S_i (at $z\to\pm\infty$), $i\neq j$, $i<j$, $i,j=1,2,3$] also varies as a function of amplitude parameters. The change in the relative separation distances $(\Delta t_{ij}=t_{ij}^{-}-t_{ij}^{+})$ can be obtained from the asymptotic expressions (38)–(43). They are found to be

$$\Delta t_{12}=\frac{\Phi^1 k_{2R}-\Phi^2 k_{1R}}{k_{1R}k_{2R}}, \tag{47a}$$

$$\Delta t_{13}=\frac{\Phi^1 k_{3R}-\Phi^3 k_{1R}}{k_{1R}k_{3R}}, \tag{47b}$$

$$\Delta t_{23}=\frac{\Phi^2 k_{3R}-\Phi^3 k_{2R}}{k_{2R}k_{3R}}, \tag{47c}$$

where Φ^j's, $j=1,2,3$, are defined in Eq. (46) and k_{jR}'s represent the real parts of k_j's.

E. Nature of collision

Now it is of interest to look into the nature of the collisions in the three-soliton interaction process, that is, whether it is pairwise or not. This can be answered from the asymptotic expressions presented in Eqs. (38)–(46). For example, let us consider soliton 1 (S_1). The net change in the amplitudes of the two modes of soliton S_1 is given by the transition amplitudes T_i^1, $i=1,2$, that is,

$$\begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix}=\begin{pmatrix} T_1^1 & 0 \\ 0 & T_2^1 \end{pmatrix}\begin{pmatrix} A_1^{1-} \\ A_2^{1-} \end{pmatrix}, \tag{48}$$

where T_1^1 and T_2^1 are defined in Eq. (45a). The above form of transition relations is obtained by expanding Eq. (44).

Let us presume first that the collision process is a pairwise one and then verify this assertion. According to our assumption $k_{1I}>k_{2I}>k_{3I}$, and so the first collision occurs between S_1 and S_2 as shown schematically in Fig. 4. Then during collision with S_2, the two modes of S_1 change their amplitudes (intensities) by $\tilde{T}_1^1$ and $\tilde{T}_2^1$, respectively. Their forms follow from the basic two-soliton interaction process discussed in Sec. V, Eq. (25b). This can be expressed in mathematical form as

$$\begin{pmatrix} \tilde{A}_1^{1+} \\ \tilde{A}_2^{1+} \end{pmatrix}=\begin{pmatrix} \tilde{T}_1^1 & 0 \\ 0 & \tilde{T}_2^1 \end{pmatrix}\begin{pmatrix} A_1^{1-} \\ A_2^{1-} \end{pmatrix}, \tag{49a}$$

where

$$\begin{pmatrix} \tilde{T}_1^1 \\ \tilde{T}_2^1 \end{pmatrix}=\begin{pmatrix} \dfrac{e^{\delta_{31}}}{\alpha_1^{(1)}} \\ \dfrac{e^{\delta_{32}}}{\alpha_1^{(2)}} \end{pmatrix} e^{-(R_4+R_2-R_1)/2}. \tag{49b}$$

Again the above expressions can be obtained straightforwardly from Eq. (25b) with $N=2$.

Now the resulting soliton ($\tilde{S}_1$), after the first collision, is allowed to collide with the third soliton (S_3) (see Fig. 4). From asymptotic expressions (38)–(45) and using above Eqs. (49), it can be shown that

$$\begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix}=\begin{pmatrix} \hat{T}_1^1 & 0 \\ 0 & \hat{T}_2^1 \end{pmatrix}\begin{pmatrix} \tilde{A}_1^{1+} \\ \tilde{A}_2^{1+} \end{pmatrix}, \tag{50a}$$

where

$$\begin{pmatrix} \hat{T}_1^1 \\ \hat{T}_2^1 \end{pmatrix}=\begin{pmatrix} e^{\tau_{31}-\delta_{31}} \\ e^{\tau_{32}-\delta_{32}} \end{pmatrix} e^{-(R_6+R_7-R_4-R_2)/2}. \tag{50b}$$

However, using Eq. (49) in Eq. (50a), we can write

$$\begin{pmatrix} A_1^{1+} \\ A_2^{1+} \end{pmatrix}=\begin{pmatrix} \hat{T}_1^1 & 0 \\ 0 & \hat{T}_2^1 \end{pmatrix}\begin{pmatrix} \tilde{T}_1^1 & 0 \\ 0 & \tilde{T}_2^1 \end{pmatrix}\begin{pmatrix} A_1^{1-} \\ A_2^{1-} \end{pmatrix} \tag{51a}$$

$$=\begin{pmatrix} \hat{T}_1^1\tilde{T}_1^1 & 0 \\ 0 & \hat{T}_2^1\tilde{T}_2^1 \end{pmatrix}\begin{pmatrix} A_1^{1-} \\ A_2^{1-} \end{pmatrix}. \tag{51b}$$

If this is the collision scenario, then the right hand sides of Eqs. (48) and (51b) should be the same, that is,

$$T_1^1=\hat{T}_1^1\tilde{T}_1^1, \tag{51c}$$

$$T_2^1=\hat{T}_2^1\tilde{T}_2^1. \tag{51d}$$

This can be easily verified to be true directly from expressions (45) and (49)–(50). In a similar fashion, for the other two-solitons also the transition matrix can be shown as a product of two matrices corresponding to two collisions, respectively.

Now let us look at the phase shifts. It is also necessary to identify whether the total phase shift acquired by each soliton during the three-soliton collision process is a result of two consecutive pairwise collisions or not. In this regard, we again focus our attention on soliton 1 (S_1) first. Let us assume the collision to be pairwise. Then one can write the

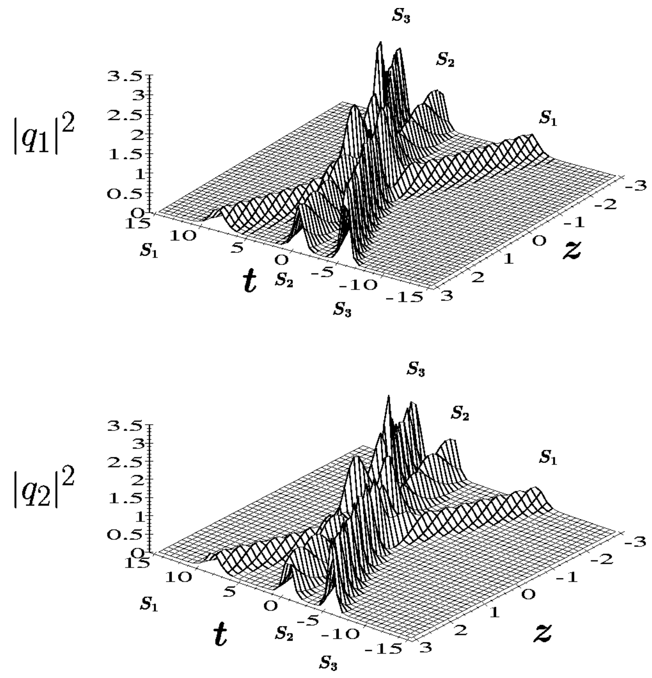


FIG. 5. Intensity profiles $|q_1|^2$ and $|q_2|^2$ of the two modes of the three-soliton solution of the 2-CNLS equations, representing elastic collision, with the parameters chosen as $k_1=1+i$, $k_2=1.5-0.5i$, $k_3=2-i$, $\alpha_1^{(1)}=\alpha_2^{(1)}=\alpha_3^{(1)}=\alpha_1^{(2)}=\alpha_2^{(2)}=\alpha_3^{(2)}=1$.

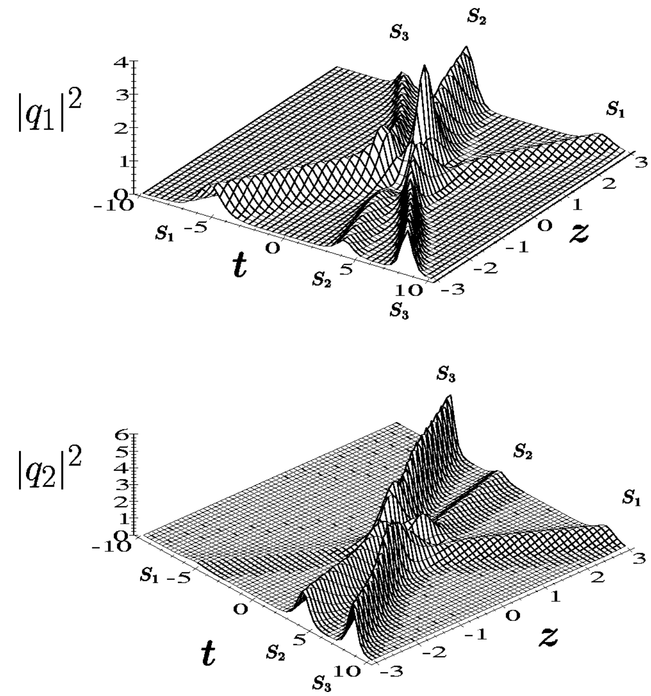


FIG. 6. Intensity profiles $|q_1|^2$ and $|q_2|^2$ of the two modes of the three-soliton solution of the 2-CNLS equations, representing the shape-changing (intensity redistribution) collision process for the choice of the parameters, $k_1=1+i$, $k_2=1.5-0.5i$, $k_3=2-i$, $\alpha_1^{(1)}=(39-80i)/89$, $\alpha_2^{(1)}=(39+80i)/89$, $\alpha_3^{(1)}=0.3+0.2i$, $\alpha_1^{(2)}=0.39$, $\alpha_2^{(2)}=\alpha_3^{(2)}=1$.

phase shift suffered by S_1 during the collision based on the analysis of the two-soliton collision process. Following Eq. (27) (with appropriately changed notations), we can write the expression for the phase shift suffered by S_1 on its collision with S_2 as

$$\tilde{\delta}=\frac{R_4-R_2-R_1}{2}. \tag{52}$$

Now the outcoming form of S_1 (which is $\hat{S}_1$) is allowed to interact with S_3 (see Fig. 4). The phase shift during this second collision can again be found from the asymptotic expressions (38)–(43) as

$$\hat{\delta}=\frac{R_7-R_6-R_4+R_2}{2}. \tag{53}$$

On the other hand, from asymptotic expressions (46), the total phase shift suffered by S_1 in a three-soliton collision process can be written as

$$\delta=\frac{R_7-R_6-R_1}{2} \tag{54}$$

$$=\tilde{\delta}+\hat{\delta}. \tag{55}$$

Thus the total phase shift suffered by the soliton 1 is the sum of the phase shifts suffered by it during pairwise collisions with soliton 2 and soliton 3, respectively. Similar conclusions can also be drawn on the phase shifts suffered by the other two-solitons as well. Thus the above analysis on the changes in the amplitudes and phase shifts during the three-soliton collision process establishes the fact that the collisions indeed occur pairwise.

It may be noted that the above results also imply that the three-soliton collision process is associative and independent of the sequence in which collisions occur, that is whether the collision occurs in the order $S_1\to S_2\to S_3$ or $S_1\to S_3\to S_2$. This property has been anticipated in the numerical study of Lewis *et al.* [26], which is now rigorously proved here.

F. Intensity redistributions and shape restoration

The asymptotic analysis not only explains the nature of the collision process, but also characterizes the collision process. It is clear from the above analysis of the three-soliton solution that in general there is an intensity redistribution among the three solitons due to pairwise interaction in all the two modes along with amplitude dependent phase shifts as in the two-soliton interaction, subject to conservation laws. We have analyzed the various three-soliton collision scenarios below.

1. Elastic collision

The standard elastic collision property of solitons results for the special case $\alpha_1^{(1)}:\alpha_2^{(1)}:\alpha_3^{(1)}=\alpha_1^{(2)}:\alpha_2^{(2)}:\alpha_3^{(2)}$. The magnitude of the transition elements $|T_j^l|, j=1,2$, and $l=1,2,3$, becomes one for this choice of parameters and there occurs no intensity redistribution among the modes except for phase shifts. This is shown in Fig. 5 for the parametric choice $\alpha_l^{(j)}=1$, $l=1,2,3$, $j=1,2$, $k_1=1+i$, $k_2=1.5-0.5i$, and $k_3=2.0-i$.

2. Shape-changing (intensity redistribution) collision

For general values of the parameters $\alpha_l^{(j)}$'s, there occurs shape-changing collisions among the three solitons, however, leaving the total intensity of each of the solitons conserved, that is, $|A_1^{l\pm}|^2+|A_2^{l\pm}|^2=1/\mu$, $l=1,2,3$. This intensity redistribution is accompanied by amplitude dependent phase shifts and changes in the relative separation distances of the solitons as discussed above. They can be calculated from expressions (38)–(47). One such shape-changing interaction is depicted in Fig. 6 for illustrative purposes. The parameters chosen are $k_1=1+i$, $k_2=1.5-0.5i$, $k_3=2-i$, $\alpha_1^{(1)}=(39-80i)/89$, $\alpha_2^{(1)}=(39+80i)/89$, $\alpha_3^{(1)}=0.3+0.2i$, $\alpha_1^{(2)}=0.39$, $\alpha_2^{(2)}=\alpha_3^{(2)}=1$. In this figure we have shown the scenario in which the three solitons in the two modes have different amplitudes (intensities) after interaction when compared to the case before interaction. Here S_1 is allowed to interact with S_2 first and then with S_3. Due to this collision, in the q_1 mode the intensity of S_1 is suppressed while that of S_2 is enhanced along with suppression of intensity in S_3. On the other hand, the reverse scenario occurs in the q_2 mode for the three solitons S_1, S_2, and S_3.

3. Shape restoration of any one of the three-solitons

The asymptotic analysis also shows that there is a possibility for any one of the three-solitons to restore its shape (amplitude or intensity) during collision. In this connection, let us look into how the shape restoring property of S_1 occurs during its collision with the other two-solitons (say S_2 and S_3). We have already shown that the collision process is a pairwise one. Then the three-soliton collision process is equivalent to two pairwise collisions. Let the first collision be parametrized by the parameters $\alpha_1^{(1)}$, $\alpha_1^{(2)}$, $\alpha_2^{(1)}$, $\alpha_2^{(2)}$, k_1, and k_2. Now we exploit the arbitrariness involved in choosing the parameters $\alpha_3^{(1)}$ and $\alpha_3^{(2)}$ in the second collision process in order to make the net transition amplitude of S_1 be unity, leaving the other two transition amplitudes of S_2 and S_3 to vary, that is,

$$T_j^1=1,\quad T_j^2\neq 1,\quad T_j^3\neq 1,\quad j=1,2. \tag{56}$$

This condition will make the soliton S_1 only be unaffected at the end of the three-soliton collision process. Then the equations corresponding to this condition are

$$A_{1R}+A_{2R}x-A_{2I}y+A_{3R}(x^2-y^2)-2A_{3I}xy+A_{4R}x+A_{4I}y+A_{5R}(x^2+y^2)+A_{6R}(x^3+xy^2)-A_{6I}(x^2y+y^3)$$
$$+A_{7R}(x^2-y^2)+2A_{7I}xy+A_{8R}(x^3+xy^2)+A_{8I}(x^2y+y^3)+A_{9R}(x^2+y^2)^2=0, \tag{57a}$$

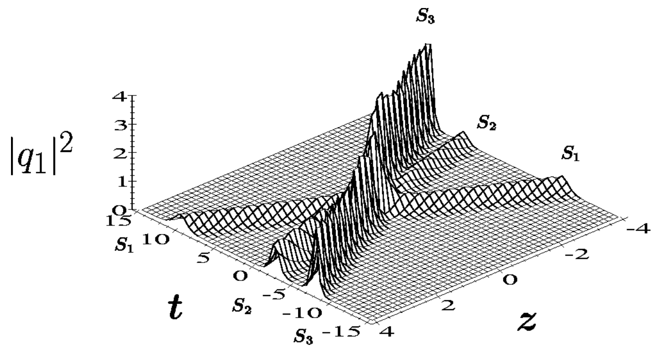
$$A_{1I}+A_{2R}y+A_{2I}x+2A_{3R}xy+A_{3I}(x^2-y^2)+A_{4I}x-A_{4R}y+A_{5I}(x^2+y^2)+A_{6I}(x^3+xy^2)+A_{6R}(x^2y+y^3)$$
$$-2A_{7R}xy+A_{7I}(x^2-y^2)-A_{8R}(x^2y+y^3)+A_{8I}(x^3+xy^2)+A_{9I}(x^2+y^2)^2=0, \tag{57b}$$

$$B_{1R}+B_{2R}x-B_{2I}y+B_{3R}(x^2-y^2)-2B_{3I}xy+B_{4R}x+B_{4I}y+B_{5R}(x^2+y^2)+B_{6R}(x^3+xy^2)-B_{6I}(x^2y+y^3)$$
$$+B_{7R}(x^2-y^2)+2B_{7I}xy+B_{8R}(x^3+xy^2)+B_{8I}(x^2y+y^3)+B_{9R}(x^2+y^2)^2=0, \tag{57c}$$

$$B_{1I}+B_{2R}y+B_{2I}x+2B_{3R}xy+B_{3I}(x^2-y^2)+B_{4I}x-B_{4R}y+B_{5I}(x^2+y^2)+B_{6I}(x^3+xy^2)+B_{6R}(x^2y+y^3)$$
$$-2B_{7R}xy+B_{7I}(x^2-y^2)-B_{8R}(x^2y+y^3)+B_{8I}(x^3+xy^2)+B_{9I}(x^2+y^2)^2=0, \tag{57d}$$

where we have taken $(\alpha_3^{(1)}/\alpha_3^{(2)})=x+iy$, the subscripts $\{lR\}$ and $\{lI\}$, $l=1,2,\ldots,9$ represent the real and imaginary parts, respectively. The expressions for the A_i's and B_i's are lengthy but can be obtained straightforwardly [by making use of Eq. (56) and expressions (45a)], and so we do not present them here. Solving these overdetermined systems of equations for x and y will give the suitable ratio $(\alpha_3^{(1)}/\alpha_3^{(2)})$, for which the shape restoring property of one of the solitons S_1 only arises in a three-soliton collision process.

Though we have not investigated the problem of the existence of solutions of Eqs. (57), one can make a numerical search and identify suitable values of x and y to demonstrate the shape restoration property. For example, in Fig. 7 with the parameters fixed at $k_1=1+i$, $k_2=1.5-0.5i$, $k_3=2-i$, $\alpha_1^{(1)}=\alpha_1^{(2)}=\alpha_2^{(2)}=1$, $\alpha_2^{(1)}=(39+80i)/89$, $\alpha_3^{(1)}=1.19$, $\alpha_3^{(2)}=(39+80i)/89$, we have demonstrated the shape restoration property. We find that while the amplitudes of the two of the solitons (S_2 and S_3) change after interaction, the amplitude



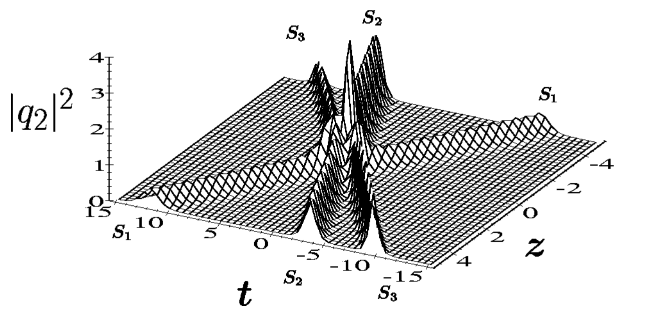


FIG. 7. Shape restoring property of soliton 1 (S_1) during its collision with the other two-solitons, soliton 2 (S_2) and soliton 3 (S_3), for the choice of parameters $k_1=1+i$, $k_2=1.5-0.5i$, $k_3=2-i$, $\alpha_1^{(1)}=\alpha_1^{(2)}=\alpha_2^{(2)}=1$, $\alpha_2^{(1)}=(39+80i)/89$, $\alpha_3^{(1)}=1.19$, $\alpha_3^{(2)}=(39+80i)/89$.

of the soliton S_1 remains unchanged during the interaction process.

In the above analysis we have required the complete restoration property of soliton S_1. However, it is also possible to require that the intensity alone be restored. In this case, condition (56) can be modified as

$$|T_j^1|=1,\ \ |T_j^2|\neq 1,\ \ |T_j^3|\neq 1,\ \ j=1,2, \tag{58}$$

leading to a set of two complicated equations for x and y (which are too lengthy to be presented here). Solving them we can find x and y. Note that the quantities x and y correspond to the real and imaginary parts of the ratio of the parameters $\alpha_3^{(1)}$ and $\alpha_3^{(2)}$, so that for every choice of x and y there exists a large set of $\alpha_3^{(1)}$ and $\alpha_3^{(2)}$ values for which shape restoration property holds good.

One might also go a step further and demand that the phase shift Φ^1 or the changes in the relative separation distances Δt_{12} and Δt_{13} vanish. These will give additional constraints on the choice of parameters $\alpha_3^{(1)}$ and $\alpha_3^{(2)}$. These considerations require separate study and we have not pursued them here. It is obvious that such shape-changing and shape restoring collision properties of the optical solitons in integrable CNLS equations, exhibiting a redistribution of intensity among the three-solitons in the two modes, will have considerable technological applications both in optical communications including wavelength division multiplexing, optical switching devices, etc., and optical computation, for example, in constructing logic gates [18,19].

G. Three-soliton solution of multicomponent CNLS equations and shape-changing collisions

The above analysis on the three-soliton collision in 2-CNLS equations can be extended straightforwardly to three-soliton solution (14) of N-CNLS equations, with arbitrary N, including $N=3$. One can identify that shape changing collision occurs here also but with a lot more possibilities for redistribution of intensities in contrast to the 2-CNLS case. The quantities characterizing the collision process here also are the intensity redistribution, amplitude dependent phase shifts, and relative separation distances between the solitons, as explained in the 2-CNLS case.

We also note that as the number of components increases from two to some arbitrary N $(N>2)$, the different possibilities for redistribution of intensity among them also increase in a manifold way. The corresponding transition matrix, measuring this redistribution, is found to be similar to Eqs. (45) with the redefinition of κ_{il}'s as given in Eq. (14b) along with the index j running from 1 to N instead of 1 to 2. The other factors, amplitude dependent phase shifts and change in relative separation distances, also bear the same form given by Eqs. (46) and (47), respectively, with this redefinition.

As to the shape restoration property one has to again solve the equations

$$T_j^1=1,\quad T_j^2\neq 0,\quad T_j^3\neq 0,\quad j=1,2,3,\ldots,N. \tag{59}$$

Alternatively for intensity restoration the conditions are

$$|T_j^1|=1,\quad |T_j^2|\neq 0,\quad |T_j^3|\neq 0,\quad j=1,2,3,\ldots,N. \tag{60}$$

Extending the above analysis, it is clear that, carrying out an asymptotic analysis of four-soliton solution given in the Appendix, it is possible to restore the shape of two of the solitons at the maximum, which can be further generalized to the arbitrary N soliton case, in which it is possible to restore the shape of $N-2$ of the solitons. We have checked in this case also from the asymptotic analysis that the soliton interaction is pairwise, and we conjecture that this should be true for the arbitrary N-soliton case as well.

VIII. MULTISOLITON SOLUTIONS AS LOGIC GATES

The state vectors and LFTs introduced in Sec. VI and the shape-changing pairwise collision nature of bright solitons mentioned in Sec. VII can be profitably used to look at the multisoliton solutions of CNLS equations as various logic gates. We believe that such an approach provides an alternative point of view of shape-changing soliton collisions to construct logic gates as discussed in Ref. [19]. The present point of view may have its own advantage as system initial conditions are chosen suitably to generate specific forms of multisolitons to represent logic gates may be much easier from a practical point of view, including replication, compared to constructing them through predetermined independent soliton collisions. In the following we will demonstrate this idea for the case of the 2-CNLS as an example.

A. Three-soliton solution and state restoration property

The shape restoration of a particular soliton in arbitrary state associated with the three-soliton solution has been discussed in Sec. VII F. Particularly, this can be well appreciated with respect to binary logic states. For example, if we consider the soliton S_1 is in "1" state with the state value $\rho^{1-}_{1,2}=1$, it implies

$$\frac{\alpha_1^{(1)}}{\alpha_1^{(2)}}=1. \tag{61}$$

To obtain this we choose $\alpha_1^{(1)}=\alpha_1^{(2)}=1$. For simplicity we require S_2 to be in the "0" state before interaction. From asymptotic expressions (39), this can be achieved by choosing the ratio $\alpha_2^{(1)}/\alpha_2^{(2)}$ as

$$\frac{\alpha_2^{(1)}}{\alpha_2^{(2)}}=\frac{k_1+k_1^*}{2k_2+k_1^*-k_1}. \tag{62}$$

Now in order to restore the state of S_1 after two collisions, we have to allow the outcome of S_1 resulting after the first collision, which may be called soliton S_1', to interact with soliton S_3 having a state inverse to the above 0 state. This state for S_3 can be identified from its asymptotic form before interaction given in Eq. (40). The resulting condition can be shown to be

$$\frac{\alpha_3^{(1)}}{\alpha_3^{(2)}}=\frac{n}{d}, \tag{63a}$$

$$n=-(\alpha_2^{(1)}+\alpha_2^{(2)})\alpha_2^{(2)*}A+\kappa_{22}(k_1-k_1^*-2k_3)B +2\alpha_2^{(2)}\alpha_2^{(2)*}C-\alpha_2^{(2)}(\alpha_2^{(1)*}+\alpha_2^{(2)*})D+|\alpha_2^{(1)}+\alpha_2^{(2)}|^2E, \tag{63b}$$

$$d=(\alpha_2^{(1)}+\alpha_2^{(2)})\alpha_2^{(1)*}A-\kappa_{22}(k_1+k_1^*)B -2\alpha_2^{(2)}\alpha_2^{(1)*}C-(\alpha_2^{(1)*}+\alpha_2^{(2)*})\alpha_2^{(2)}D, \tag{63c}$$

where

$$A=(k_1+k_1^*)(k_3+k_1^*)(k_1+k_2^*), \tag{63d}$$

$$B=(k_2+k_1^*)(k_3+k_2^*)(k_1+k_2^*), \tag{63e}$$

$$C=(k_2+k_1^*)(k_3+k_1^*)(k_1+k_2^*), \tag{63f}$$

$$D=(k_2+k_1^*)(k_3+k_2^*)(k_1+k_1^*), \tag{63g}$$

$$E=(k_3+k_2^*)(k_1+k_1^*)(k_3+k_1^*). \tag{63h}$$

In the above equation choosing the parameters satisfying conditions (61) and (62) one can fix $(\alpha_3^{(1)}/\alpha_3^{(2)})$ suitably in order to restore the state of soliton S_1. Thus the three-soliton solution given by Eq. (10) having the specific choice of parameters specified by Eqs. (61)–(63) corresponds to the state restoration of soliton S_1.

B. Four-soliton solution and COPY gate

Extending the above procedure, we can now consider the four-soliton solution given in the Appendix, and identify it as (i) a COPY gate or (ii) a ONE gate or (iii) a NOT gate studied in Ref. [19] for suitable choices of the arbitrary parameters. As an example, let us consider copying 1 state of S_1 to the output state of soliton S_4. This requires the following steps.

(1) We consider the four-soliton collision process in which the soliton S_1 collides with the soliton S_2 first and then with the soliton S_3 and finally with the soliton S_4. This sequence of collision follows from the condition $k_{1I}>k_{2I}>k_{3I}>k_{4I}$.

(2) Consider for convenience S_1 to be in the 0 state, the so-called actuator state [19]. This requires $\alpha_1^{(1)}/\alpha_1^{(2)}=0$, which can be obtained by choosing $\alpha_1^{(1)}=0$ and $\alpha_1^{(2)}$ as arbitrary.

(3) Assign 1 state to soliton S_2 before interaction, for which we need

$$\frac{\alpha_2^{(1)}}{\alpha_2^{(2)}}=\frac{k_2-k_1}{k_2+k_1^*}. \tag{64}$$

(4) After its collision with S_2 as a result of shape-changing collision the outcoming state of S_1 (say S_1') will be altered.

(5) Now let us allow the third soliton in the four-soliton solution to interact with S_1' which changes the state S_1' to S_1''.

(6) Finally, S_4 is allowed to interact with S_1''. From the asymptotic analysis, we identify the state of soliton S_4 after interaction as

$$\rho^{4+}_{1,2}=\frac{\alpha_4^{(1)}}{\alpha_4^{(2)}}. \tag{65}$$

We impose the condition on this state that this should be in the state of S_2 before interaction. Thus the parameters $\alpha_4^{(1)}$ and $\alpha_4^{(2)}$ of soliton S_4 get fixed depending upon the input state of S_2.

(7) The asymptotic analysis of the four-soliton solution given in the Appendix results in the following condition for S_4 to be in one state after interaction:

$$\frac{T_1^4A_1^{4-}}{T_2^4A_2^{4-}}=1, \tag{66}$$

where T_1^4 and T_2^4 are the transition elements of S_4 in the modes q_1 and q_2, respectively. Here $A_1^{4-}k_{4R}$ and $A_2^{4-}k_{4R}$ are the amplitudes of soliton S_4 before interaction in the two modes, respectively.

(8) If we flip the input state of S_2 from 1 to 0 state by suitably choosing the $\rho_{1,2}^{2-}$'s parameters then the condition on soliton S_4's output will become

$$\frac{T_1^4 A_1^{4-}}{T_2^4 A_2^{4-}}=0. \tag{67}$$

(9) In the above two Eqs. (66) and (67) only free parameters are $\alpha_3^{(1)}$ and $\alpha_3^{(2)}$. In principle, we can solve these two complex equations to obtain the free complex parameters $\alpha_3^{(1)}$ and $\alpha_3^{(2)}$. Then for the given choice of parameters the state of the incoming soliton S_2 can be copied on to the outgoing soliton S_4.

Thus a four-soliton collision process with the above premise is equivalent to a COPY gate. A similar procedure can be extended to other gates mentioned above as well. One can extend this idea further to identify a FANOUT gate from a five-soliton solution. It appears that one can pursue the idea ultimately to identify the NAND gate itself as a multisoliton solution following the construction of Steiglitz in Ref. [19]. Fuller details will be reported elsewhere.

IX. BRIGHT SOLITON SOLUTIONS AND PARTIALLY COHERENT SOLITONS

As mentioned in the Introduction, the recent observations by several authors [11,12,25] have shown that N-CNLS equations (1) can support N-PCSs solutions. In general, these PCSs are said to be special cases of the so-called multisoliton complexes [2] which are nonlinear superposition of fundamental bright solitons. It has also been demonstrated that these PCSs are formed only if the number of components in Eq. (1) is equal to the number of solitons. Then it is quite natural to look for the 2-PCS, 3-PCS, 4-PCS, etc., as special cases of the two-soliton solution of the 2-CNLS, three-soliton solution of the 3-CNLS, four-soliton solution of the 4-CNLS equations, etc., respectively, deduced in Secs. III and IV. In the following, we indeed show that the PCSs reported in Refs. [11,12,25] result as special cases, that is, specific choices of some of the arbitrary complex parameters, from the bright soliton solutions of CNLS equations discussed in Secs. III and IV, thereby showing the origin of the various interesting properties of the PCS solutions.

A. 2-PCS : A special case of the bright two-soliton solution of 2-CNLS equations

Let us consider the stationary limit of the two-soliton solution of the 2-CNLS equations (Manakov system) given by Eq. (8), that is, $k_{nI}=0$, for the special choice of the parameters, $\alpha_1^{(1)}=e^{\eta_{10}}$, $\alpha_2^{(2)}=-e^{\eta_{20}}$, $\alpha_1^{(2)}=-\alpha_2^{(1)}=0$, where η_{j0}'s, $j=1,2$, are now restricted as real constants. Then Eq. (8) becomes

$$q_1=\left(e^{\eta_1}+\frac{\mu(k_{1R}-k_{2R})e^{\eta_1+\eta_2+\eta_2^*}}{4k_{2R}^2(k_{1R}+k_{2R})}\right)\bigg/\tilde{D}, \tag{68a}$$

$$q_2=\left(-e^{\eta_2}+\frac{\mu(k_{1R}-k_{2R})e^{\eta_1+\eta_1^*+\eta_2}}{4k_{1R}^2(k_{1R}+k_{2R})}\right)\bigg/\tilde{D}, \tag{68b}$$

where

$$\tilde{D}=1+\mu\left[\frac{e^{\eta_1+\eta_1^*}}{4k_{1R}^2}+\frac{e^{\eta_2+\eta_2^*}}{4k_{2R}^2}\right]+\frac{\mu^2(k_{1R}-k_{2R})^2e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}}{16k_{1R}^2k_{2R}^2(k_{1R}+k_{2R})^2} \tag{68c}$$

and

$$\eta_j=k_{jR}(t+ik_{jR}z)+\eta_{j0},\quad j=1,2. \tag{68d}$$

This stationary solution can be easily identified as the 2-PCS expression (13)–(15) given in Ref. [12] with the identification of $\overline{x_j}$'s as $\overline{t_j}$'s, $j=1,2$,

$$\overline{t_1}=t-t_1=t+\frac{\eta_{10}}{k_{1R}}+\frac{1}{2k_{1R}}\ln\left[\frac{\mu(k_{1R}-k_{2R})}{4k_{1R}^2(k_{1R}+k_{2R})}\right], \tag{69a}$$

$$\overline{t_2}=t-t_2=t+\frac{\eta_{20}}{k_{2R}}+\frac{1}{2k_{2R}}\ln\left[\frac{\mu(k_{1R}-k_{2R})}{4k_{2R}^2(k_{1R}+k_{2R})}\right]. \tag{69b}$$

As the 2-PCS is a special case of the bright two-soliton solution of 2-CNLS equations, it is also characterized by $\alpha_i^{(j)}$'s (through η_{j0}'s) and k_{iR}'s resulting in amplitude dependent phases, and hence amplitude dependent relative separation distances. To be specific, in the PCSs the change in the relative separation distance plays a predominant role in determining their shape as pointed out in Refs. [11,12]. These PCSs can be classified into two types as symmetric and asymmetric depending on the relative separation distances. Defining the relative separation distance $t_{12}=t_2-t_1$, one can check that, for $t_{12}=0$, the PCS bears a symmetric form with respect to its propagation direction and is known as symmetric PCS [11]. It takes an asymmetric form for $t_{12}\neq 0$ and is known as asymmetric PCS [11]. From Eqs. (69), the relative separation distances for the stationary 2-PCS can be obtained as

$$t_{12}=t_2-t_1=\frac{\eta_{10}}{k_{1R}}-\frac{\eta_{20}}{k_{2R}}+\frac{1}{2k_{1R}}\ln\left[\frac{\mu(k_{1R}-k_{2R})}{4k_{1R}^2(k_{1R}+k_{2R})}\right]-\frac{1}{2k_{2R}}\ln\left[\frac{\mu(k_{1R}-k_{2R})}{4k_{2R}^2(k_{1R}+k_{2R})}\right]. \tag{70}$$

Typical forms of symmetric and asymmetric stationary 2-PCS are shown in Fig. 8, which similar to those in Ref. [12].

B. 3-PCS: A special case of the bright three-soliton solution of 3-CNLS equations

Since it has been observed that the PCS solutions exist when the number of components is equal to the number of solitons propagating in the system, we consider next the three-soliton solution of the 3-CNLS equations in order to show that the 3-PCS is a special case of the three-soliton solution. Thus considering the stationary limit $k_{nI}=0$, $n=1,2,3$, of the three-soliton solution of 3-CNLS equations given by Eq. (14) with $N=3$, and making the following parametric choice:

$$\alpha_1^{(1)}=e^{\eta_{10}}, \quad \alpha_2^{(2)}=-e^{\eta_{20}}, \quad \alpha_3^{(3)}=e^{\eta_{30}},$$

$$\alpha_1^{(2)}=\alpha_1^{(3)}=\alpha_2^{(1)}=\alpha_2^{(3)}=\alpha_3^{(1)}=\alpha_3^{(2)}=0, \tag{71}$$

where η_{j0}'s, $j=1,2,3$, are restricted to real parameters, we obtain

$$q_1=\left[e^{\eta_1}+\frac{\mu(k_{1R}-k_{2R})e^{\eta_1+\eta_2+\eta_2^*}}{4k_{2R}^2(k_{1R}+k_{2R})}+\frac{\mu(k_{1R}-k_{3R})e^{\eta_1+\eta_3+\eta_3^*}}{4k_{3R}^2(k_{1R}+k_{3R})}\right.$$
$$\left.+\frac{\mu^2(k_{2R}-k_{1R})(k_{3R}-k_{1R})(k_{3R}-k_{2R})^2e^{\eta_3+\eta_3^*+\eta_2+\eta_2^*+\eta_1}}{16k_{2R}^2k_{3R}^2(k_{2R}+k_{1R})(k_{3R}+k_{1R})(k_{3R}+k_{2R})^2}\right]\Big/\widetilde{D_1}, \tag{72a}$$

$$q_2=\left[-e^{\eta_2}+\frac{\mu(k_{1R}-k_{2R})e^{\eta_1+\eta_1^*+\eta_2}}{4k_{1R}^2(k_{1R}+k_{2R})}+\frac{\mu(k_{3R}-k_{2R})e^{\eta_3+\eta_3^*+\eta_2}}{4k_{3R}^2(k_{3R}+k_{2R})}\right.$$
$$\left.+\frac{\mu^2(k_{2R}-k_{1R})(k_{3R}-k_{2R})(k_{3R}-k_{1R})^2e^{\eta_3+\eta_3^*+\eta_1+\eta_1^*+\eta_2}}{16k_{1R}^2k_{3R}^2(k_{2R}+k_{1R})(k_{3R}+k_{2R})(k_{3R}+k_{1R})^2}\right]\Big/\widetilde{D_1}, \tag{72b}$$

$$q_3=\left[e^{\eta_3}+\frac{\mu(k_{3R}-k_{1R})e^{\eta_1+\eta_1^*+\eta_3}}{4k_{1R}^2(k_{1R}+k_{3R})}+\frac{\mu(k_{3R}-k_{2R})e^{\eta_2+\eta_2^*+\eta_3}}{4k_{2R}^2(k_{3R}+k_{2R})}\right.$$
$$\left.+\frac{\mu^2(k_{3R}-k_{1R})(k_{3R}-k_{2R})(k_{2R}-k_{1R})^2e^{\eta_2+\eta_2^*+\eta_1+\eta_1^*+\eta_3}}{16k_{1R}^2k_{2R}^2(k_{3R}+k_{1R})(k_{3R}+k_{2R})(k_{2R}+k_{1R})^2}\right]\Big/\widetilde{D_1}. \tag{72c}$$

Here,

$$\widetilde{D_1}=1+\mu\left[\frac{e^{\eta_1+\eta_1^*}}{4k_{1R}^2}+\frac{e^{\eta_2+\eta_2^*}}{4k_{2R}^2}+\frac{e^{\eta_3+\eta_3^*}}{4k_{3R}^2}\right]+\frac{\mu^2(k_{1R}-k_{2R})^2e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*}}{16k_{1R}^2k_{2R}^2(k_{1R}+k_{2R})^2}+\frac{\mu^2(k_{1R}-k_{3R})^2e^{\eta_1+\eta_1^*+\eta_3+\eta_3^*}}{16k_{1R}^2k_{3R}^2(k_{1R}+k_{3R})^2}$$
$$+\frac{\mu^2(k_{3R}-k_{2R})^2e^{\eta_2+\eta_2^*+\eta_3+\eta_3^*}}{16k_{2R}^2k_{3R}^2(k_{2R}+k_{3R})^2}+\left[\frac{\mu^3(k_{2R}-k_{1R})^2(k_{3R}-k_{1R})^2(k_{3R}-k_{2R})^2e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\eta_3+\eta_3^*}}{64k_{1R}^2k_{2R}^2k_{3R}^2(k_{1R}+k_{2R})^2(k_{1R}+k_{3R})^2(k_{2R}+k_{3R})^2}\right]. \tag{72d}$$

The above solution can be easily rewritten as Eqs. (16)–(18) for the 3-PCS case given in Ref. [12]. As in the case of 2-PCS, here also we identify $\overline{x_j}$'s given in Ref. [12] as $\overline{t_j}$'s, $j=1,2,3$, which are defined below as

$$\overline{t_1}=t-t_1=t+\frac{\eta_{10}}{k_{1R}}+\frac{1}{2k_{1R}}\ln\left[\frac{\mu(k_{2R}-k_{1R})(k_{3R}-k_{1R})}{4k_{1R}^2(k_{1R}+k_{2R})(k_{1R}+k_{3R})}\right], \tag{73a}$$

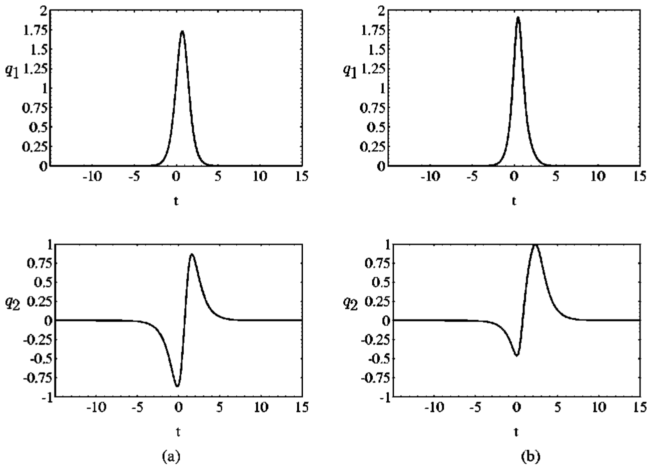


FIG. 8. Typical 2-PCS forms for the Manakov system for $z=0$, see Eqs. (68), with $k_1=1.0$ and $k_2=2.0$: (a) symmetric case $(t_{12}=0)$, (b) asymmetric case $(t_{12}=1)$.

$$\overline{t_2}=t-t_2=t+\frac{\eta_{20}}{k_{2R}}+\frac{1}{2k_{2R}}\ln\left[\frac{\mu(k_{2R}-k_{1R})(k_{3R}-k_{2R})}{4k_{2R}^2(k_{1R}+k_{2R})(k_{2R}+k_{3R})}\right], \tag{73b}$$

$$\overline{t_3}=t-t_3=t+\frac{\eta_{30}}{k_{3R}}+\frac{1}{2k_{3R}}\ln\left[\frac{\mu(k_{3R}-k_{1R})(k_{3R}-k_{2R})}{4k_{3R}^2(k_{1R}+k_{3R})(k_{2R}+k_{3R})}\right]. \tag{73c}$$

These 3-PCSs can also be classified as symmetric and asymmetric as in the case of 2-PCSs. The stationary 3-PCS is symmetric when $t_{12}=t_{13}=0$ and asymmetric otherwise. In Fig. 9 we have shown the symmetric and asymmetric 3-PCS solutions.

C. 4-PCS: A special case of the four-soliton solution of 4-CNLS equations

In a similar fashion as in the above two cases, the four-soliton solution of the 4-CNLS equations given in the Appendix with $N=4$ can also be shown to reduce to 4-PCS given by Eqs. (19)–(23) in Ref. [12] by choosing $k_{nI}=0$, $\alpha_1^{(1)}=e^{\eta_{10}}$, $\alpha_2^{(2)}=-e^{\eta_{20}}$, $\alpha_3^{(3)}=e^{\eta_{30}}$, $\alpha_4^{(4)}=-e^{\eta_{40}}$, $\alpha_i^{(j)}=0, i,j=1,2,3,4$, $i\neq j$. Since it is straightforward but lengthy to write down the form, we desist from presenting the solution here. Here the t_j's, $j=1,2,3,4$, are defined as

$$t_1=-\frac{\eta_{10}}{k_{1R}}-\frac{1}{2k_{1R}}\ln\left[\frac{\mu(k_{2R}-k_{1R})(k_{3R}-k_{1R})(k_{4R}-k_{1R})}{4k_{1R}^2(k_{2R}+k_{1R})(k_{3R}+k_{1R})(k_{4R}+k_{1R})}\right], \tag{74a}$$

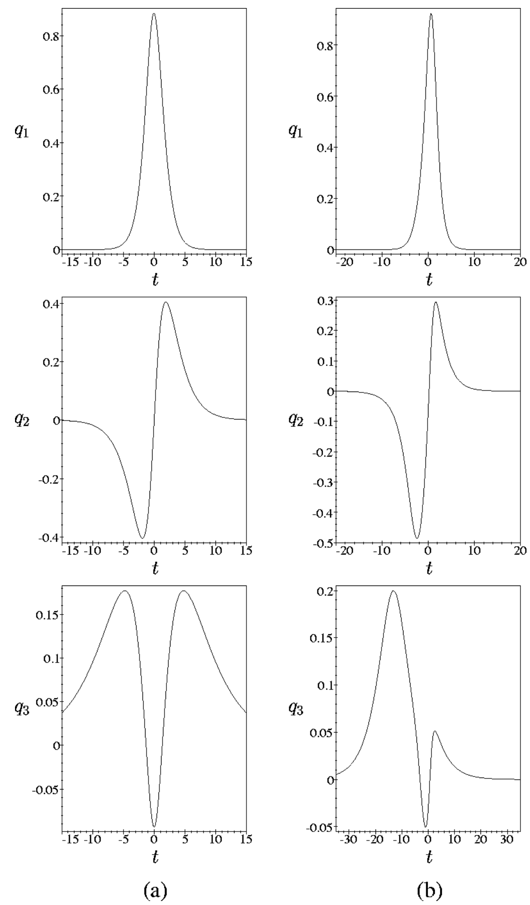


FIG. 9. Typical 3-PCS forms for the integrable 3-CNLS system for $z=0$ with $k_1=1.0$, $k_2=0.5$, and $k_3=0.2$, see Eqs. (72). (a) Symmetric case $(t_{12}=t_{13}=0)$, (b) asymmetric case $(t_{12}=1,\ t_{13}=2)$.

$$t_2=-\frac{\eta_{20}}{k_{2R}}-\frac{1}{2k_{2R}}\ln\left[\frac{\mu(k_{2R}-k_{1R})(k_{3R}-k_{2R})(k_{4R}-k_{2R})}{4k_{2R}^2(k_{2R}+k_{1R})(k_{3R}+k_{2R})(k_{4R}+k_{2R})}\right], \tag{74b}$$

$$t_3=-\frac{\eta_{30}}{k_{3R}}-\frac{1}{2k_{3R}}\ln\left[\frac{\mu(k_{3R}-k_{1R})(k_{3R}-k_{2R})(k_{4R}-k_{3R})}{4k_{3R}^2(k_{3R}+k_{1R})(k_{3R}+k_{2R})(k_{4R}+k_{3R})}\right], \tag{74c}$$

$$t_4=-\frac{\eta_{40}}{k_{4R}}-\frac{1}{2k_{4R}}\ln\left[\frac{\mu(k_{4R}-k_{1R})(k_{4R}-k_{2R})(k_{4R}-k_{3R})}{4k_{4R}^2(k_{4R}+k_{1R})(k_{4R}+k_{2R})(k_{4R}+k_{3R})}\right]. \tag{74d}$$

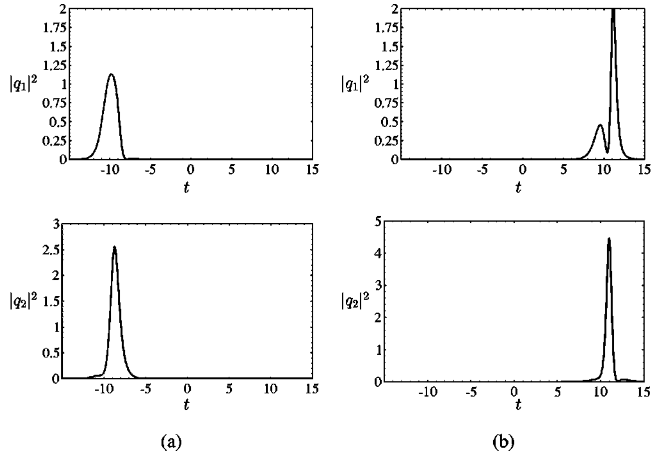


FIG. 10. Intensity profiles showing the collision scenario of two 1-PCSs, with equal velocities, at (a) $z=-5$ and (b) $z=5$, given by special choice of parameters (as given in text) in the two-soliton solution of the Manakov system.

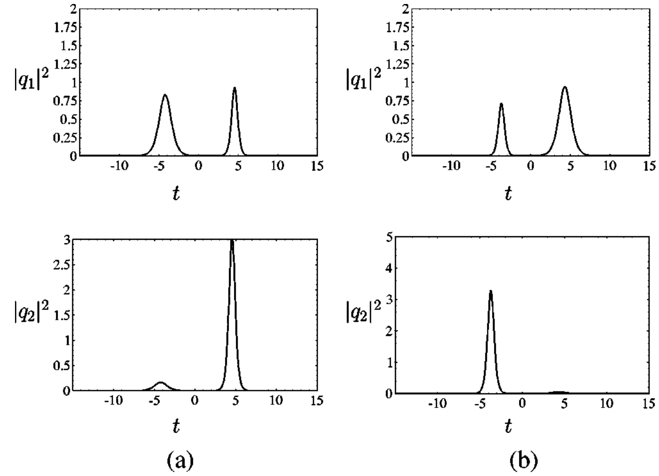


FIG. 11. Intensity profiles showing the collision scenario of two 1-PCSs, moving with equal but opposite velocities, at (a) $z=-5$ and (b) $z=5$, given by special choice of parameters (as given in text) in the two-soliton solution of the Manakov system.

Here also the symmetric PCS results for $t_{ij}=0$, $j>i$, $i,j=1,2,3,4$, and asymmetric PCS for $t_{ij}\neq 0$, $j>i$.

Extending this idea to arbitrary *N*, it is clear that the *N*-PCS is a special case of the *N*-soliton solution of *N*-CNLS equations (1). It has been noticed in Refs. [11,12] that these PCSs are of variable shape. The reason for the variable shape can be traced naturally to the nontrivial dependence of phases on the complex parameters $\alpha_i^{(j)}$'s as shown above. Thus it is clear that any change in the amplitude will affect the phase part of the solitons and vice versa. Since we have explicitly shown that *N*-PCSs are special cases of bright *N*-soliton solutions of *N*-CNLS equations, they possess variable shape as a consequence of the shape dependence on the $\alpha_i^{(j)}$ parameters.

D. Propagation of partially coherent solitons and their collision properties

The intriguing collision properties of the partially coherent solitons reported in Refs. [11,12] can be well understood by writing down the expression for PCSs with nonvanishing k_{nI}'s, that is nonstationary special cases of multicomponent higher-order bright soliton solutions discussed in Secs. III and IV. For the nonstationary PCSs we can choose as a special case the complex parameters $\alpha_i^{(j)}$'s $(i\neq j)$ to be functions of k_{nI}'s such that they vanish as $k_{nI}=0$. As we make these $k_{nI}\neq 0$, then $\alpha_i^{(j)}$'s $(i\neq j)$ also vary, thereby making the collision scenario interesting. We can consider both the cases of equal and unequal velocities, which exhibit similar behaviors.

As a first example, we consider the propagation of the 2-PCS comprising two solitons with equal velocities $(k_{1I}=k_{2I})$ in PR media. Its propagation can be studied by choosing (for illustrative purposes) $\alpha_1^{(2)}=k_{1I}$ and $\alpha_2^{(1)}=(0.25+1.02i)k_{2I}$ as functions of velocities $(k_{jI}, j=1,2)$ such that they vanish when $k_{jI}=0$, $j=1,2$. This is shown in Fig. 10 for the parameters $\alpha_1^{(1)}=2.0+i$, $\alpha_2^{(2)}=1$, $k_1=1.0+i$, and $k_2=2.0+i$. For the unequal velocity case $(k_{1I}\neq k_{2I})$, the PCS collision is shown in Fig. 11 for the parametric choice $\alpha_1^{(1)}=1.0$, $\alpha_1^{(2)}=k_{1I}$, $\alpha_2^{(1)}=-[(22+80i)/89]k_{2I}$, $\alpha_2^{(2)}=-2.0$, $k_1=1.0+i$, and $k_2=2.0-i$. This can also be viewed as the collision of two 1-PCS which are spread among the two components, which are traveling with equal but opposite velocities.

Now let us consider the collision of 2-PCS and 1-PCS in PR media. This is equivalent to the three-soliton collision in the 3-CNLS system with specific choice of parameters. We consider the case, in which the complex parameters $\alpha_1^{(2)}, \alpha_1^{(3)}$, $\alpha_2^{(1)}$, $\alpha_2^{(3)}$, $\alpha_3^{(1)}$, $\alpha_3^{(2)}$ are nonvanishing and as functions of k_{nI}'s, $n=1,2,3$. Then the resulting asymptotic forms of the 3-PCS propagation is shown in Fig. 12 for the parametric choice $\alpha_1^{(1)}=1.0$, $\alpha_1^{(2)}=\alpha_1^{(3)}=k_{1I}$, $\alpha_2^{(1)}=-0.5k_{2I}$, $\alpha_2^{(2)}=0.25$, $\alpha_2^{(3)}=0.02k_{2I}$, $\alpha_3^{(1)}=-[(22+80i)/89]k_{3I}$, $\alpha_3^{(2)}=2k_{3I}$, $\alpha_3^{(3)}=-2$, $k_1=1.0+i$, $k_2=1.5-i$, and $k_3=2.0-i$. In the above figures it can be verified that the total intensity of the individual solitons comprising the PCS is conserved.

The above analysis on PCS propagation clearly shows that, there will be a variation in the shape of the PCS during its collision with other PCSs. The explanation for this result follows from the shape-changing (intensity redistribution) nature of fundamental bright soliton collision of the integrable CNLS equations, explained in Sec. V. Further, we have also observed that the collision of two PCSs each comprising *m* and *n* soliton complexes, respectively, such that $m+n=N$ studied in Refs. [11,12,25], is equivalent to the interaction of *N* fundamental bright solitons (for suitable specific choice of parameters) represented by the special case of *N*-soliton solution of the *N*-CNLS system. It should also be noted that in the collision process the total intensity of individual solitons comprising the *N*-PCS is conserved. This is due to the complete integrable nature of the *N*-CNLS equations (1).

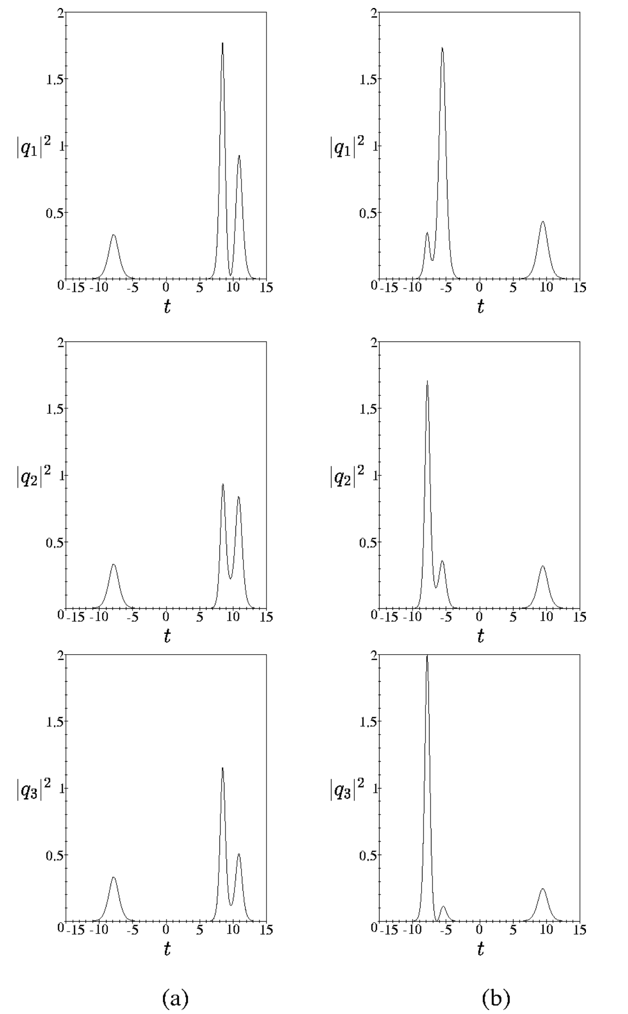


FIG. 12. Intensity profiles showing the collision scenario of 2-PCS with 1-PCS at (a) $z=-4$ and (b) $z=4$ given by special choice of parameters (as given in text) in the three-soliton solution of Eq. (1) with $N=3$.

E. Multisoliton complexes

In the above we have considered the CNLS equations with number of components (say p) is equal to the number of fundamental solitons (say q). This is only a special case of the multisoliton complexes and has been much discusssed recently. However, the results are scarce for the case $p\neq q$, except for the work of Sukhorukov and Akhmediev [25], where the incoherent soliton collision is demonstrated numerically. To elucidate the understanding we present a form of the three-soliton complex in which three solitons are spread among the two components, by suitably choosing the parameters in the explicit expression, Eq. (10). This has been shown in Fig. 13 with the parameters chosen as $\alpha_1^{(1)}=\alpha_1^{(2)}=1.0$, $\alpha_2^{(1)}=0.5,\alpha_2^{(2)}=0.25$, $\alpha_3^{(1)}=(22+80i)/89$, $\alpha_3^{(2)}=-2$, $k_1=1.0+i$, and $k_2=1.5+i$. From the figure and the analysis of the soliton interaction, it is clear that the shapes of these complexes strongly depend on the $\alpha_i^{(j)}$'s along with k_j's which determine how the solitons are spread up among the components. For the same case there exist various forms of multisoliton complexes depending on the spreading up of

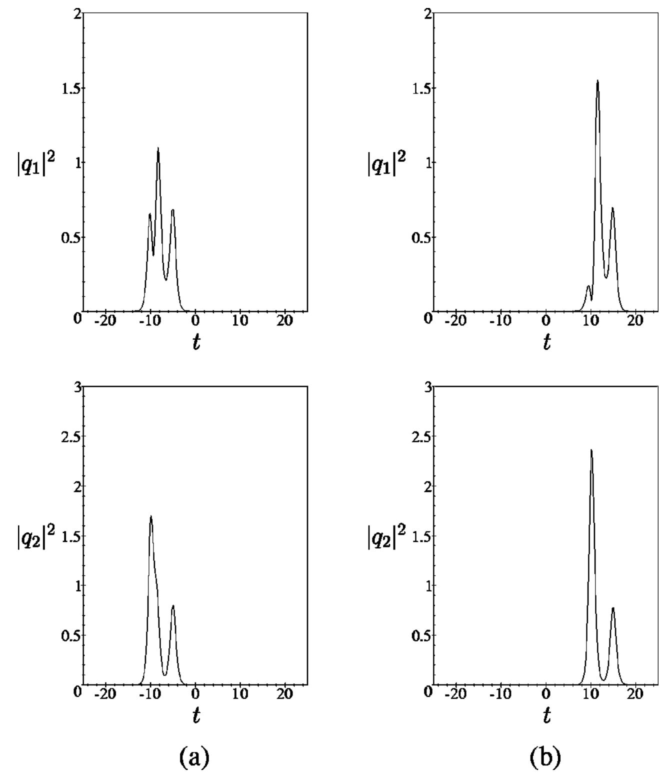


FIG. 13. Intensity profiles of a multisoliton complex comprising three solitons spread up in two components propagating in photorefractive media: a special case of three-soliton solution (10) of the integrable 2-CNLS system for the parameters chosen as in the text, (a) at $z=-5$ and (b) at $z=5$.

solitons in the two components. As a consequence of this, multisoliton complexes will possess a rich variety of structures in comparison with the PCSs.

X. CONCLUSION

We conclude this paper by stating that the collision processes of solitons in coupled nonlinear Schrödinger equations lead to very many exciting different properties and potential applications. The different properties include shape-changing intensity redistributions, amplitude dependent phase shifts, and relative separation distances, within the pairwise collision mechanism of solitons. Interestingly, it is identified that the intensity redistribution characterizing the shape-changing collision process in N-CNLS equations can be written as a generalized linear fraction transformation. This will give further impetus in constructing multistate logic, multi-input logic gates, memory storage devices, and so on, by using soliton interactions. The implication of these properties requires further deep investigations. Further, viewing the recently much discussed objects, multisoliton complexes, partially coherent solitons as special cases of the bright soliton solution enhances the understanding of their various properties. We expect the interaction study presented here will shine more light on spatial soliton propagation in $(1+1)$D photorefractive planar waveguides.

ACKNOWLEDGMENTS

The work reported here was supported by the Department of Science and Technology, Goverment of India, and the Council of Scientific and Industrial Research (CSIR), Government of India, in the form of research projects to M.L. T.K. wishes to acknowledge CSIR for financial support.

APPENDIX: FOUR-SOLITON SOLUTION

In this appendix for completeness, we present the form of the four-soliton solution of the 2-CNLS equations by generalizing the two- and three-soliton solutions of it, which can be obtained by terminating the power series as

$$g^{(j)}=\chi g_1^{(j)}+\chi^3 g_3^{(j)}+\chi^5 g_5^{(j)}+\chi^7 g_7^{(j)}, \tag{A1a}$$

$$f=1+\chi^2 f_2+\chi^4 f_4+\chi^6 f_6+\chi^8 f_8, \tag{A1b}$$

and solving the resulting set of linear partial differential equations recursively. It can be written as

$$q_s=\frac{N^{(s)}}{D}, \quad s=1,2, \tag{A2a}$$

where

$$\begin{aligned}
N^{(s)}=&\sum_{i=1}^{4}\alpha_i^{(s)}e^{\eta_i}+\left(\frac{1}{2}\right)\sum_{\substack{i,j,l=1\\(i\neq l)}}^{4}\frac{(k_l-k_i)(\alpha_l^{(s)}\kappa_{ij}-\alpha_i^{(s)}\kappa_{lj})}{(k_j^*+k_i)(k_j^*+k_l)}e^{\eta_i+\eta_j^*+\eta_l}\\
&+\left(\frac{1}{12}\right)\sum_{\substack{i,j,l,\\m,n=1\\(i\neq l\neq n;\\j\neq m)}}^{4}\left[\frac{(k_n-k_i)(k_l-k_i)(k_l-k_n)(k_m^*-k_j^*)}{(k_j^*+k_i)(k_j^*+k_l)(k_j^*+k_n)(k_m^*+k_i)(k_m^*+k_l)(k_m^*+k_n)}\right]\\
&\times\{\alpha_i^{(s)}[\kappa_{lm}\kappa_{nj}-\kappa_{lj}\kappa_{nm}]+\alpha_n^{(s)}[\kappa_{lj}\kappa_{im}-\kappa_{ij}\kappa_{lm}]+\alpha_l^{(s)}[\kappa_{nm}\kappa_{ij}-\kappa_{im}\kappa_{nj}]\}e^{\eta_i+\eta_j^*+\eta_l+\eta_m^*+\eta_n}\\
&-\left(\frac{1}{144}\right)\sum_{\substack{i,j,l,m,\\n,o,p=1\\(i\neq l\neq n\neq p;\\j\neq m\neq o)}}^{4}\frac{1}{D_1}[(k_p-k_i)(k_p-k_l)(k_p-k_n)(k_n-k_l)(k_n-k_i)(k_l-k_i)(k_o^*-k_m^*)(k_o^*-k_j^*)(k_m^*-k_j^*)]\\
&\times\begin{vmatrix}\alpha_i^{(s)} & \alpha_l^{(s)} & \alpha_n^{(s)} & \alpha_p^{(s)}\\ \kappa_{ij} & \kappa_{lj} & \kappa_{nj} & \kappa_{pj}\\ \kappa_{im} & \kappa_{lm} & \kappa_{nm} & \kappa_{pm}\\ \kappa_{io} & \kappa_{lo} & \kappa_{no} & \kappa_{po}\end{vmatrix}e^{\eta_i+\eta_j^*+\eta_l+\eta_m^*+\eta_n+\eta_o^*+\eta_p},
\end{aligned} \tag{A2b}$$

where

$$\eta_i=k_i(t+ik_iz), \quad i=1,2,3,4, \tag{A2c}$$

$$\begin{aligned}
D_1=&(k_j^*+k_i)(k_j^*+k_l)(k_j^*+k_n)(k_j^*+k_p)\\
&\times(k_m^*+k_i)(k_m^*+k_l)(k_m^*+k_n)(k_m^*+k_p)\\
&\times(k_o^*+k_i)(k_o^*+k_l)(k_o^*+k_n)(k_o^*+k_p)
\end{aligned} \tag{A2d}$$

and

$$D=1+\sum_{i,j=1}^{4}\frac{\kappa_{ij}}{k_i+k_j^*}e^{\eta_i+\eta_j^*}+\left(\frac{1}{4}\right)\sum_{\substack{i,j,l,m=1\\(i\neq l;j\neq m)}}^{4}\frac{(k_l-k_i)(k_m^*-k_j^*)(\kappa_{ij}\kappa_{lm}-\kappa_{im}\kappa_{lj})}{(k_j^*+k_i)(k_j^*+k_l)(k_m^*+k_i)(k_m^*+k_l)}e^{\eta_i+\eta_j^*+\eta_l+\eta_m^*}$$
$$+\left(\frac{1}{36}\right)\sum_{\substack{i,j,l,\\m,n,o=1\\(i\neq l\neq n;\\j\neq m\neq o)}}^{4}\frac{(k_n-k_l)(k_n-k_i)(k_l-k_i)(k_o^*-k_m^*)(k_o^*-k_j^*)(k_m^*-k_j^*)}{D_2}$$
$$\times\begin{vmatrix}\kappa_{ij} & \kappa_{im} & \kappa_{io}\\ \kappa_{lj} & \kappa_{lm} & \kappa_{lo}\\ \kappa_{nj} & \kappa_{nm} & \kappa_{no}\end{vmatrix}e^{\eta_i+\eta_j^*+\eta_l+\eta_m^*+\eta_n+\eta_o^*}+\frac{|k_1-k_2|^2|k_2-k_3|^2|k_3-k_1|^2|k_4-k_1|^2|k_2-k_4|^2|k_3-k_4|^2}{\prod_{i=1}^{4}(k_i+k_i^*)|k_1+k_2^*|^2|k_1+k_3^*|^2|k_1+k_4^*|^2|k_2+k_3^*|^2||k_2+k_4^*|^2|k_3+k_4^*|^2}$$
$$\times\begin{vmatrix}\kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14}\\ \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24}\\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{34}\\ \kappa_{41} & \kappa_{42} & \kappa_{43} & \kappa_{44}\end{vmatrix}e^{(\eta_1+\eta_1^*+\eta_2+\eta_2^*+\eta_3+\eta_3^*+\eta_4+\eta_4^*)}. \tag{A2e}$$

In the above

$$D_2=(k_j^*+k_i)(k_j^*+k_l)(k_j^*+k_n)(k_m^*+k_i)(k_m^*+k_l)(k_m^*+k_n)(k_o^*+k_i)(k_o^*+k_l)(k_o^*+k_n) \tag{A2f}$$

and

$$\kappa_{il}=\frac{\mu(\alpha_i^{(1)}\alpha_l^{(1)*}+\alpha_i^{(2)}\alpha_l^{(2)*})}{(k_i+k_l^*)},\quad i,l=1,2,3,4. \tag{A2g}$$

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